## HELIANTUS

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## MATHEMATICS <br> for <br> PRIMARY, MIDDLE and HIGH SCHOOL <br> Tadeusz STYŠ

Zasanie 0.1 The little shepherd saw flying storks and shouted, I think, there are 100 of them. The older shepherd replied much less, if there were twice as many, and a half as many, and a quarter as many, and if you were to fly with them, then 100 of them would be with you. How many storks were flying. in the sky?


Picture Joseph Chetmoński (1849-1914). Storks
Solution
$\underbrace{2 * \text { more of them }}_{8 * \text { quarters }}+\underbrace{\text { half of them more }}_{+2 * \text { quarters }}+\underbrace{\text { quarter of them more }+1}_{+1 * \text { quarter }+1}=\underbrace{100}_{=100}$
$11 *$ quarters $=99, \quad 1$ quartererc $=\frac{99}{11}=9$,
1 quarteer $=\frac{1}{4} *$ of them
The number of storks $=4 *$ quarters $=4 * 9=36$.
Answer : The number of storks is 36

### 0.1 From the author

The script "Mathematics for Primary School and Secondary School" was developed on the basis of several decades of work and the author's experience in various educational systems, mainly in universities, but also in primary and secondary schools in Poland and Africa. Thus, this text is not a textbook for primary school. However, as a comprehensive material, it covers the content of mathematics in the basic and extended program of the scholar at the second and third stage of education. This compilation of the entirety of elementary and high school mathematics can be particularly helpful as material for individual study.

Tadeusz STYŚ

### 0.2 Introduction

The material presented in the script "Mathematics for Primary and Secondary Schools and High Schools..." exceeds the core curriculum of mathematics taught in primary schools and largely includes topics in the mathematics curriculum taught in high schools . Naturally, this extended range of subjects allows the selection of advanced subjects with a degree of difficulty at the level of primary school students with greater predispositions and interests in science subjects. In the script, the reader will find many interesting algorithms, theorems with proofs, and examples outside of the basic math curriculum at the second and third stage of learning.

A systematic description of natural, rational, irrational, algebraic and real numbers and the properties of arithmetic operations on these numbers are given in Chapters 1 and 2.

Arithmetic and algebraic expressions as basic concepts related to algorithms and their coding in programming languages are described and clarified by many examples in the chapter 3.

Numerous examples dealing with simple and compound percentages are given in Chapter 4. Chapter 5 covers basic knowledge about arithmetic and geometric sequences and series. Prime numbers and their properties make up the most important and interesting content of Chapter 6. It mainly covers the following topics: Fundamental Theorem of Arithmetic, Euclid's Algorithm, greatest common divisor $\operatorname{GCD}(\mathrm{a}, \mathrm{b})$ and least common multiple $\mathrm{LCM}(\mathrm{a}, \mathrm{b})$ of natural numbers $a, b$.
The representation of numbers in computers in floating decimal point and arithmetic operations on numbers with a finite number of digits with the analysis of absolute and relative errors are described in Chapter 7..

The divisibility features of integers and the division with remainder congruence supported by numerous examples constitute the content of Chapter 8 .

The general principle of creating numbers in binary, decimal, octal, and other natural-base $\rho>1$ systems is described in detail in Chapter 9.
Chapter 10 is devoted to polynomials, the simplest class of functions of fundamental importance in mathematics and its applications.
The simplified multiplication formulas, Newton's binomial and Pascal's triangle with numerous examples are described in Chapter 11.
Chapters $12,13,14,15$, and 14 are devoted to elementary functions which include linear functions, rational functions, square root functions, exponential and logarithmic functions.

The 17 -th chapter is devoted to plane geometry - planimetry. The scope of planar geometry includes structures with a ruler and compasses for flat figures, and unitary relationships in triangles, rectangles, parallelograms, circles, and regular polygons.
Chapter 18, Spatial Geometry- Stereometry, describes the following topics:

1. Cartesian coordinate system. Points and vectors in space.
2. Parametric line equation
3. Prisms and rectangulars, volume and surface area
4. Pyramids, volume and surface area
5. Solids of rotation: cylinder, sphere, cone, included and area.

Among the spatial solids we distinguish regular and Platonic solids. Regular solids have all congruent faces. Platonic solids, which include only the tetrahedron, cube, octahedron, dodecahedron, and icosahedron, were considered in ancient times by the Platonic Academy (427-343 B.C.) to be ideal figures.
Chapter 20 is devoted to trigonometry of knowledge about the measurement relationships between sides and angles in triangles. An important part of this chapter is the characterization and analysis of trigonometric functions defined in a right triangle and on the trigonometric circle.

Chapter 19 of Combinatorics is very interesting and important in applications, mainly in computer science. Combinatorics includes concepts such as factorial $n$ ! of a natural number n , permutations in n -element sets, combinations and variations in n -element sets.

Chapter 21, descriptive statistics, important in primary education, describes concepts such as statistics, diagrams, correlations, means and medians, standard deviations, and variances. The calculus of probability deals with the study of the laws governing random phenomena, i.e. those phenomena whose course or result cannot be unequivocally predicted. Random events and their probability of occurrence are explained in detail in chapter 22.

### 0.3 Greek Mathematics

For a thousand years B.C. during the Greek Empire, the Ancient Greeks assimilated the achievements of many Middle Eastern and Indian cultures in the fields of Astronomy, Architecture, Medicine, Mathematics and Physics. The Greeks became the best teachers leaving behind well-documented Mathematics and Science literature. An important part of them activity was the organization of the Schools of Philosophy, Mathematics and Science in Greece, Egypt and Mesopotamia.
Tales of Miletus (625-545 B.C.)

founded the first Ionian School of Astronomy, Mathematics and Philosophy. Pythagoras (569-500B.C.) from Samos

founded a co-educational Mystical School of Philosophy and Mathematics in the city of Kroton on the Ionian Sea. Pythagoras, a music lover, created the foundations for determining the pitch of sounds, author of the Pythagorean Theorem on measurement relationships in a right triangle and triangles of Pythagorean numbers $a, b, c$

$$
a^{2}+b^{2}=c^{2}
$$

Euclid (330-275 B.C.) Dean of the Department of Arithmetic and Geometry at the University of Alexandria (330-275 B.C.E.) has gone down in history as one of the greatest ancient mathematicians.
Author of the books Elements of Arithmetic and Geometry. Euclid's geometry is still taught
in primary and secondary schools.

## Euklides



Euclid (330-275 B.C.)

Archimedes (287-212 B.C.) the son of the astronomer from Syracus announced the well known Law of Archimedes, formulated the basis for the calculus of the infinitesimal. In the Middle Ages, Newton (1642-1727) and Leibnitz (1646-1716) developed the idea of an infinitely small calculus. The results of their research on the bill infinitely small had a significant impact on the further development of mathematics and science. Namely, Newton and Leibnitz created the foundations of differential and integral calculus. Calculus, is taught at polytechnics and universities as a compulsory subject.


Archimedes (287-212 B.C.)

Many other Greeks of merit entered the history of science for good. Among them, Plato (429-428 B.C.), the founder of idealistic philosophy, and Aristotle (384-322 B.C.), a student of Plato and a teacher of Alexander the Great.
Plato founded the famous Platonic Academy in Athens. After Plato's death, Aristotle
founded his own high school in 343 B.C.

(Plato and Aristotle)
Let's mention Socrates (469-399 B.C.), the father of philosophy and a lover of mathematics, who has been proclaimed the teacher of all time.


Socrates was a historical figure. Socrates did not leave any writings. All we know about him are the accounts of his students: Plato and Xenophon, as well as the accounts of Aristotle and Greek historians

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## Chapter 1

## Natural numbers and integers

### 1.1 Introduction

The concept of natural numbers and simple arithmetic operations have been known since about 50,000 years ago. We know this on the basis of archaeological and historical discoveries.
On the other hand, the first systematic description of natural number arithmetic was developed by ancient Greeks in the Ionian school of Thales ( $625-545 \mathrm{BC}$ ), in the Pythagorean school (569-475 BC), on the university of Alexandria by Euclid (330-267 BC) and by Archmedes of Syracuse (287-212 BC)
Number theory is still an inspiring subject of numerous works published in the leading writings of number theory. In the last few decades, wide applications of number theory have been observed in the design of computer systems in cryptography and data protection, and in the creation of new algorithms for administration and social programs.

### 1.2 Natural numbers

The set of positive natural numbers is denoted by the symbol

$$
\begin{equation*}
N_{+}=\{1,2,3, \ldots, n, \ldots\} \tag{1.1}
\end{equation*}
$$

By convention, zero is included in the set of natural numbers. Then we denote the set of natural numbers with the symbol


Numeric exes. Natural numbers

### 1.2.1 Properties of natural numbers

Obvious properties of the sets of natural nambers $N_{+}$and $N$.

The set of natural numbers $N_{+}$is the subset of natural numbers $N$, we write $N_{+} \subset N$. The sum of the natural numbers $m+n$ is also a natural number. Thus, for any natural numbers $m, n \in N$ their sum

$$
m+n \in N
$$

belongs to the set of natural numbers.
It means that the set of natural numbers is closed due to addition operations.
For example, for the numbers $m=7, n=5$, their sum

$$
m+n=7+5=12 \in N
$$

12 is a natural number.
The addition operation is commutative for any natural numbers $m, n$ sum

$$
m+n=n+m
$$

For example, $5+3=3+5=8 \in N$.
Similarly, the set of natural numbers is closed to multiplication operations and the multiplication operation is commutative
Namely, the product of natural numbers $m * n$ is a natural number.
So for any natural numbers $m, n \in N$ their product

$$
m * n \in N
$$

belongs to the set of natural numbers.
That is, the set of natural numbers is closed due to multiplication operations.
For example, for natural numbers $m=7, n=5$, their product

$$
m * n=7 * 5=35 \in N
$$

is a natural number. The multiplication operation is commutative for any natural numbers $m, n$ product

$$
m * n=n * m
$$

However, the result of subtracting natural numbers is not always a natural number.
For example, number difference

$$
3-5
$$

is not a natural number, but the difference $3-5=-2$ is an integer. We will discuss integers in the next section.

### 1.2.2 Examples

Example 1.1 Calculate the sum of 10 consecutive natural numbers

$$
S_{10}=1+2+3+4+5+6+7+8+9+10
$$

using only one multiplication operation and one division operation.

## Solution:

Let's write the sums in reverse order and add the equal sides as follows:

$$
\begin{array}{ccc}
S_{10} & = & 1+2+3+4+5+6+7+8+9+10 \\
S_{10} & = & 10+9+8+7+6+5+4+3+2+1 \\
--- & \cdots & ----------------- \\
2 * S_{10} & = & \underbrace{11+11+11+11+11+11+11+11+11+11}_{10 \text { ingredients sums }}
\end{array}
$$

Where do we calculate the sum of $S_{10}$ using one multiplication and one division.

$$
S_{10}=10 * 11: 2=55
$$

Example 1.2 What is the general formula for the sum of $n$ successive natural numbers

$$
S_{n}=1+2+3+\cdots+n
$$

Give an example of the application of this formula using only one multiplication operation and one division operation.

## Solution:

Let's write the components of the sum in reverse order and add the equal sides as follows:

$$
\begin{array}{ccc}
S_{n} & = & 1+2+3+\cdots+(n-2)+(n-1)+n \\
S_{n} & = & n+(n-1)+(n-2)+\cdots+3+2+1 \\
--- & \cdots & ----------------- \\
2 * S_{n} & = & \underbrace{(n+1)+(n+1)+(n+1)+\cdots+(n+1)+(n+1)}_{n \text { ingredients sums }}
\end{array}
$$

Where do we calculate the sum of $S_{n}$.

$$
S_{n}=\frac{n(n+1)}{2}
$$

For $n=10$, we calculate $S_{10}$

$$
S_{10}=\frac{10 * 11}{2}=55
$$

### 1.3 Integers

As we know, in the set of natural numbers, the subtraction operation is not always feasible. For example, there is no natural number that would be the result of subtracting 9 from 5 because the difference

$$
5-9
$$

is not a natural number.
Negative numbers.
Opposing numbers are two numbers lying on the number line at the same distance from zero but on opposite sides of zero.
Negative numbers have the property that their sum is 0 .
Thus, $-m$ is the opposite of $m$ then

$$
-m+m=0
$$

For example

$$
\text { for } m=7, \text { number opposite }-m=-7, \quad \text { then } 7+(-7)=0
$$

Likewise

$$
\text { for } m=-9, \text { number opposite }-m=9, \quad \text { then }-9+9=0 .
$$

On the number line we have natural numbers marked on the right side of zero, and on the left side of zero we have marked opposite numbers to natural numbers.


Integers are marked below on the number line


All natural numbers and all opposite to them form the set of integers
We denote the set of integers with the letter $C$, write

$$
C=\{\ldots \ldots \ldots \ldots \ldots \ldots \ldots-5,-4,-3,-2,-1,0,1,2,3,4,5 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots\}
$$

Example 1.1 On the number line, mark the numbers that are opposite to the given natural numbers


Below, on the number line, we see natural numbers

$$
0,1,2,3,4,5
$$

and opposite to natural nmbers

$$
0,-1,-2,-3,-4,-5
$$



### 1.3.1 Examples and questions

Example 1.2 The difference between zero and successive natural numbers

$$
\begin{aligned}
& 0-1=-1, \quad 0-6=-6 \\
& 0-2=-2, \quad 0-7=-7 \\
& 0-3=-3, \\
& 0-8=-8 \\
& 0-4=-4,
\end{aligned} 0-9=-9010=-10
$$

Question 1.1 Calculate the difference

$$
\begin{array}{lll}
0-11 \\
0-12 \\
0-13 \\
0-14 & , & 0-16 \\
0-15 & , & 0-17 \\
0-19 \\
0-20
\end{array}
$$

Example 1.3 Check the difference

$$
\begin{aligned}
& 5-10=-5,10-16=-6 \\
& 6-12=-6,11-17=-7 \\
& 7-13=-6,12-18=-6 \\
& 8-14=-5,13-19=-6 \\
& 9-15=-6,14-20=-6
\end{aligned}
$$

Question 1.2 Calculate the difference

$$
\begin{array}{lll}
1-10 \\
3-12 \\
5-14 \\
7-15 & , & 10-20 \\
9-16 & , & 12-22 \\
\hline
\end{array} \quad 13-23-24-24
$$

### 1.4 Addition and subtraction of integers

The set of integers

$$
C=\{\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots-5,-4,-3,-2,-1,0,1,2,3,4,5 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots\}
$$

is closed for addition and subtraction operations. That is, for any two integers $m, n \in C$ sum

$$
m+n \in C
$$

and the difference

$$
m-n \in C
$$

of these numbers is an integer.

### 1.4.1 Examples

Let us consider some examples of adding and subtracting integers.
Adding a negative integer ntoaninteger m , subtracting the opposite number $-n$ from $m$

$$
\begin{array}{rllll}
5+(-4) & =5-4 & =1, & \text { for } m=5, & n=-4 \\
-5+(-4) & =-5-4 & =-9, & \text { for } m=-5, & n=-4 \\
8-(-7) & =8+7=15, & \text { for } m=8, & n=-2 \\
-8-(-7) & =-8+7 & =-1, & \text { for } m=-8 & n=-7
\end{array}
$$

### 1.5 Multiplication of integers

The set of integers

$$
C=\{\ldots \ldots \ldots \ldots \ldots \ldots \ldots-5,-4,-3,-2,-1,0,1,2,3,4,5 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots\}
$$

is closed of with respect to multiplication operation. That is, for any two integers $m, n \in C$ product

$$
m * n \in C
$$

of these numbers is an integer.
The product of $m * n>0$ of integers $m, n \in C$ is a positive integer if $m>0, n>0$ are positive or $m<0, n<0$ are negative, otherwise the product of $m * n<0$ of the numbers $m>0, n<0$ or $m<0, n>0$ is negative.

## Example 1.4

$$
\begin{array}{ll}
4 * 8 & =32, \quad \text { for } \quad m=4>0, \quad n=8>0 \\
(-6) *(-7) & =42, \quad \text { for } \quad m=-6<0, n=-7<0
\end{array}
$$

### 1.6 Division of integers

Note that the result of dividing integers is not always an integer $l$.
For example

$$
9: 2=4 \frac{1}{2}, \quad \text { but } \quad 9: 3=3
$$

Thus, the set of integers is not closed due to the division operation.
The quotient of $m: n>0$ integers total $m$ and $n \neq 0$ is positive, if if both numbers are positive or both are negative, otherwise if $m>0, n<0$ or $m<0, n>0$ is the quotient of $m: n<0$ is negative.

Example 1.5 The result of dividing dw ó positive numbers is positive. Namely, let $m=8$ and $n=4$

$$
m: n=8: 4=2
$$

Example 1.6 The result of dividing two negative numbers is positive. Let $m=-8$ and $n=-4$. Then

$$
m: n=(-8):(-4)=2
$$

Similarly, dividing a positive number by negative is negative. The result of dividing a negative number by positive is negative.

## Example 1.7

$$
\begin{array}{llll}
(-8): 4 & =-2, & \text { for } \quad m=-8, & n=4 \\
8:(-4) & =-2, & \text { for } & m=8,
\end{array} \quad n=-4
$$

Question 1.3 Calculate the value of the expression of arithmetic expression
(i) $\quad(-8: 4+14: 7)-(9: 3-6: 2)$
(ii) $(-18): 3+12: 3-(15:(-5)-(16: 2))$
(iii) $((-24): 6+12: 3)-(15:(-5)-(16: 2))$

### 1.7 Even and odd numbers

The set of natural numbers consists of two disjoint subsets from the subset of even numbers and the subset of odd numbers. We write
Even numbers

$$
n=2 k \text { for } k=0,1,2,3, \ldots
$$

So we have an infinite sequence of even numbers

$$
0,2,4,6,8,10,12, \ldots ;
$$

Odd numbers. We write odd numbers

$$
n=2 k+1, \quad \text { for } k=0,1,2,3, \ldots
$$

So we have an infinite sequence of odd numbers

$$
1,3,5,7,9,11,13, \ldots
$$

Note that even numbers are divided by 2 , while odd numbers are divided by 2 with the remainder of 1 .

### 1.7.1 Examples

Example 1.3 The sum of three consecutive even numbers is 84. Find these numbers.

## Solution:

The consecutive even numbers are

$$
2 n-2, \quad 2 n, \quad 2 n+2,
$$

Their sum

$$
(2 n-2)+2 n+(2 n+2)=6 n=84
$$

We calculate n :

$$
\begin{aligned}
& 2 n-2=2 * 14-2=26 \\
& 2 n=2 * 14=28 \\
& 2 n+2=2 * 14+2=30
\end{aligned}
$$

Check: We calculate the sum of three consecutive even numbers

$$
26+28+30=84
$$

Example 1.4 How many different two-digit odd numbers can be made from the digits $1,2,3$. ?

## Solution:

Odd numbers made up of digits $1,2,3$ have unity digits 1 or 3
All two digit odd numbers and different, which have unity digits 1 or 3 , we write in the table

| 11 | 21 | 31 |
| :--- | :--- | :--- |
| 13 | 23 | 33 |

From the table, we read that there are $2 * 3=6$ different odd numbers formed from the digits $1,2,3$.

Example 1.5 How many different three digit odd numbers can be made from the digits $1,2,3$. ?

## Solution:

Odd numbers made up of $1,2,3$ have unity digits 1 or 3
All three digit odd different numbers that have unity digit 1 or 3 , we write in the table

| 111 | 211 | 311 |
| :--- | :--- | :--- |
| 113 | 213 | 313 |
| 121 | 221 | 321 |
| 123 | 223 | 323 |
| 131 | 231 | 331 |
| 133 | 233 | 333 |

From the table, we read that there are $6 * 3=18$ different odd numbers formed from the digits $1,2,3$.

Example 1.6 How many different two-digit even numbers can be made from 1, 2, 3, 4, 5, 6, 7?

## Solution:

Even numbers made of $1,2,3,4,5,6,7$ have three unity digits
2 or 4 or 6
Let's write all the different two-digit even numbers that have the units digit 2 or 4 or 6

| 12 | 14 | 16 |
| :--- | :--- | :--- |
| 22 | 24 | 26 |
| 32 | 34 | 36 |
| 42 | 44 | 46 |
| 52 | 54 | 56 |
| 62 | 64 | 66 |
| 72 | 74 | 76 |

From the table, we read that there are $3 * 7=21$ different even numbers formed from the digits $2,4,6$.

Example 1.7 The sum of three consecutive odd numbers is 51. Find these numbers.
Solution: The consecutive odd numbers are

$$
2 n+1, \quad 2 n+3, \quad 2 n+5
$$

From the equation

$$
(2 n+1)+(2 n+3)+(2 n+5)=6 n+9=51 .
$$

we find $n$

$$
6 n+9=51, \quad 6 n=42, \quad n=42: 6=7
$$

Let's calculate three consecutive odd numbers

$$
\begin{aligned}
& 2 n+1=2 * 7+1=15 \\
& 2 n+3=2 * 7+3=17 \\
& 2 n+5=2 * 7+5=19
\end{aligned}
$$

Indeed, the sum of three consecutive odd numbers

$$
15+17+19=51
$$

Example 1.8 Find the sum of 10 consecutive even numbers

$$
S_{10}=2+4+6+8+10+12+14+16+18+20
$$

Solution: Let's write the components of the sum in reverse order and add the equal sides as follows:

$$
\begin{array}{ccc}
S_{10} & = & 2+4++6+8+10+12+14+16+18+20 \\
S_{10} & = & 20+18+16+14+12+10+8+6+4+2 \\
--- & \cdots & ---------------- \\
2 * S_{10} & =\underbrace{22+22+22+22+22+22+22+22+22+22}_{10 \text { components sum }}
\end{array}
$$

Hence, we find the sum of $S_{10}$ using one multiplication and one division.

$$
S_{10}=10 * 22: 2=110 \quad \text { or } \quad \mathrm{S}_{10}=\frac{10 * 22}{2}=110
$$

Example 1.9 What is the general formula for the sum of $n$ consecutive even numbers

$$
S_{n}=2+4+\cdots+(2 n-2)+2 n
$$

Give an example of this formula using only one multiplication operation and one division operation.

Solution: Let's write the sums in reverse order and add the equal sides as follows:

$$
\begin{array}{cccccccc}
S_{n} & = & 2+ & 4+ & 6+ & \cdots+ & 2 n-2+ & 2 n \\
S_{n} & = & 2 n+ & (2 n-2)+ & (2 n-4)+ & \cdots+ & 4+ & 2 \\
--- & \cdots & --- & --- & --- & \cdots & --- & --- \\
2 * S_{n} & = & (2 n+2)+ & (2 n+2)+ & (2 n+2)+ & \cdots+ & (2 n+2)+ & (2 n+2)
\end{array}
$$



Hence, we find the sum of $S_{n}$.

$$
S_{n}=\frac{n(2 n+2)}{2}=\frac{2 n(n+1)}{2}=n(n+1)
$$

For $n=10$, we calculate $S_{10}$

$$
S_{10}=\frac{10 * 22}{2}=10 * 11=110
$$

Example 1.10 Find the sum of 10 consecutive odd numbers

$$
S_{10}=1+3+5+7+9+11+13+15+17+19
$$

Solution: Let's write the sums in reverse order and add the equality side by side as follows:

$$
\begin{array}{ccc}
S_{10} & = & 1+3+5+7+9+11+13+15+17+19 \\
S_{10} & = & 19+17+15+13+11+9+7+5+3+1 \\
--- & \cdots & ----------------- \\
2 * S_{10} & = & \underbrace{20+20+20+20+20+20+20+20+20+20}_{10 \text { components of the sum }}
\end{array}
$$

We find the sum of $S_{10}$ using one multiplication and one division.

$$
S_{10}=10 * 20: 2=100 \quad \text { or } \quad \mathrm{S}_{19}=\frac{10 * 20}{2}=100
$$

Example 1.11 What is the general formula for the sum of $n$ consecutive odd numbers

$$
S_{n}=1+3+\cdots+(2 n-3)+(2 n-1)
$$

Give an example of this formula using only one multiplication operation and one division operation.

Solution: Let's write the components of the sum in reverse order and add them with equal sides

$$
\begin{array}{cccccccc}
S_{n} & = & 1+ & 3+ & 5+ & \cdots+ & (2 n-3)+ & (2 n-1) \\
S_{n} & = & (2 n-1)+ & (2 n-3)+ & (2 n-5)+ & \cdots+ & 3+ & 1 \\
--- & \cdots & --- & --- & --- & \cdots & --- & --- \\
2 * S_{n} & = & 2 n+ & 2 n+ & 2 n+ & \cdots+ & 2 n+ & 2 n
\end{array}
$$

Hence we find the sum $S_{n}$.

$$
2 S_{n}=n * 2 n, \quad S_{n}=\frac{n * 2 n}{2}=n * n=n^{2}
$$

For $n=10$, we calculate $S_{10}$

$$
S_{10}=10 * 10=100
$$

Example 1.12 Prove that the algebraic expression

$$
a^{2}+(a+2)(a+2)+(a+4)(a+4)+1
$$

is divisible by 12 for every odd number of $a$.

## Solution:

Since the number a is odd, then for cerain $n$

$$
a=2 * n-1
$$

Substituting

$$
a=2 * n-1
$$

to the algebraic expression, we will get

$$
\begin{aligned}
& a^{2}+(a+2)(a+2)+(a+4)(a+4)+1= \\
= & (2 * n-1)(2 * n-1)+(2 * n-1+2)(2 * n-1+2)+ \\
+ & 2 * n-1+4)(2 * n-1+4)+1= \\
= & (4 * n * n-4 * n+1)+(2 * n+1(2 * n+1)+ \\
+ & (2 * n+3)(2 * n+3)+1= \\
= & \left(4 * n^{2}-4 * n+1\right)+\left(4 * n^{2}+4 * n+1\right)+\left(4 * n^{2}+12 * n+9\right)= \\
= & 12 * n^{2}+12 * n+12= \\
= & 12 *\left(n^{2}+n+1\right)
\end{aligned}
$$

For every odd number $a=2 * n-1$, this expression factories 12 times $\left(n^{2}+n+1\right)$. So this algebraic expression is divisible by 12 for each odd value of $a$.

### 1.7.2 Questions

Question 1.4 How many different two-digit even numbers can be made from 1, 2, 3.?
Question 1.5 How many different three-digit even numbers can be made from the digits $1,2,3$.?

Question 1.6 How many different three-digit odd numbers can be made from $1,2,3,4,5,6,7$ ?
Question 1.7 Calculate the sum of 15 consecutive natural numbers

$$
S_{15}=1+2+3+4+5+6+7+8+9+10+11+12+13+14+15
$$

using only one multiplication operation and one division operation.
Question 1.8 Calculate the sum of 9 consecutive natural numbers starting with 10

$$
S_{9}=10+12+13+14+15+16+17+18+19
$$

using the formula for the sum of $n$ consecutive natural numbers.
Question 1.9 The sum of three consecutive natural numbers is 45. Find these numbers.
Question 1.10 The sum of three consecutive even numbers is 120. Find these numbers.
Question 1.11 The sum of three consecutive odd numbers is 180. Find these numbers.
Question 1.12 Prove that the value of an algebraic expression

$$
n^{2}+n+1
$$

is an odd number for any natural $n=0,1,2,3, \ldots$;

### 1.8 Exponential operation

Multiplying a number by itself several times, we calculate its power. For example, if we multiply the number 2 , we get its next powers

$$
\begin{array}{ll}
2^{0} & =1 \\
2^{1} & =2 \\
2 * 2 & =2^{2}=4 \\
2 * 2 * 2 & =2^{3}=8 \\
2 * 2 * 2 * 2 & =2^{4}=16
\end{array}
$$

Similarly, if we multiply the numbers 3 by ourselves, we get the next powers

$$
\begin{array}{ll}
3^{0} & =1 \\
3^{1} & =3 \\
3 * 3 & =3^{2}=9 \\
3 * 3 * 3 & =3^{3}=27 \\
3 * 3 * 3 * 3 & =3^{4}=81 \\
3 * 3 * 3 * 3 * 3 & =3^{5}=243
\end{array}
$$

Each number $a \neq 0$ raised to the power of 0 is equal to 1 .
For example

$$
1^{0}=1, \quad 5^{0}=1, \quad 6^{0}=1, \quad 7^{0}=1, \quad 14^{0}=1, \quad 259^{0}=1
$$

Generally, we write $n_{t h}$ product of the number $a \neq 0$ in the following form

$$
\begin{array}{ll}
a^{0}=1, & 2^{0}=1 \\
\underbrace{a * a \ldots * a}_{n-\text { factors }}=a^{n}, & \underbrace{2 * 2 \ldots * 2}_{n-\text { factors }}=2^{n}
\end{array}
$$

So, we call $a$ the base with the exponent $n$ of powr $a^{n}$.

Question 1.13 Calculate the powers

$$
\begin{array}{llll}
4^{0}= & , & 4^{1}= & , 4^{2}= \\
5^{2}= & , & 5^{3}= & , 5^{4}= \\
10^{2}= & , & 10^{3}= & , 10^{4}=
\end{array}
$$

Arithmetic operations on powers. The following operations are possible on powers

1. Multiply powers with the same base

$$
a^{p} * a^{q}=a^{p+q}
$$

for any $p, q$.
For example, for $a=2, p=3, q=5$ we have

$$
2^{3} * 2^{5}=2^{3+5}=2^{8}=256
$$

2. Dividing powers with the same base

$$
\frac{a^{p}}{a^{q}}=a^{p-q}
$$

for any numbers $p, q$.
For example, for $a=2, p=5, q=3$ we have

$$
2^{5}: 2^{3}=2^{5-3}=2^{2}=4
$$

3. Exponentiation of powers with the same base

$$
\left(a^{p}\right)^{q}=a^{p * q}
$$

for any $p, q$.
For example, for $a=2, p=2, q=3$ we have

$$
\left(2^{3}\right)^{2}=2^{2 * 3}=2^{6}=64
$$

4. Power of the product of numbers with the same exponent

$$
(a * b)^{n}=a^{n} * b^{n}
$$

is equal to the product of power g .
For example, for $a=2, b=3, n=3$, wehave

$$
(2 * 3)^{3}=2^{3} * 3^{3}=8 * 27=216
$$

5. Power of the quotient of numbers with the same exponent

$$
\left(\frac{a}{b}\right)^{n}=\frac{a^{n}}{b^{n}}
$$

equal to the power quotient.
For example, for $a=4, b=2, n=3$ we have

$$
(4: 2)^{3}=4^{3}: 2^{3}=64: 8=8 \quad \text { or } \quad\left(\frac{4}{2}\right)^{3}=\frac{4^{3}}{2^{3}}=\frac{64}{8}=8
$$

Example 1.13 Calculate the value of an arithmetic expression

$$
\frac{2^{3} * 3^{4}}{2^{2} * 3^{3}}
$$

Solution. Performing operations on powers let's calculate

$$
\frac{2^{3} * 3^{4}}{2^{2} * 3^{3}}=2 * 3=6
$$

Question 1.14 Calculate the value of an arithmetic expression
(i) $5^{2} * 2^{3}+3^{2} * 2^{3}-4^{2} * 5^{2}$
(ii) $\frac{2^{3} * 3^{2}+5^{2} * 7^{2}-2 * 6 * 8-1}{3^{2} * 5^{2}-2^{3} * 4^{2}+3}$

Answer (ii): 12
Question 1.15 Calculate the value of an arithmetic expression

$$
\frac{3^{3} * 2^{3}-3^{2} * 2^{2}}{3 * 2^{3}+2 * 3}
$$

Re: 6

### 1.9 Numbers divisible by $2,3,4$ and 5

- Numbers that are divisible by 2

$$
0,2,4,6,8,10,12,14,16,18,20,22,24, \ldots ;
$$

we write in general form

$$
n=2 k, \quad \text { for } \quad k=0,1,2,3, \ldots
$$

## Example 1.8

$$
\begin{aligned}
& 124: 2=62, \quad \text { or } \quad \frac{124}{2}=62 \\
& 316: 2=158, \quad \text { or } \quad \frac{2528}{2}=1264
\end{aligned}
$$

- Numbers divisible by 3

$$
0,3,6,9,12,15,18,21,24,27, \ldots
$$

we write in general form

$$
n=3 k, \quad \text { for } \quad k=0,1,2,3, \ldots
$$

Clearly, the numbers of the form $3 * k, \quad k=0,1,2,3, \ldots$; are divisible by 3 , when

$$
3 * k: 3=k, \quad \text { or } \quad \frac{3 * k}{3}=k
$$

for each natural $k=0,1,2,3,4, \ldots$;
Generally, the number $n$ is divisible by 3 if and only if the sum of the digits of $n$ divides by 3.

Example $1.9 n=54$ is divisible by 3 because its sum of the digits $5+4=9$ is divisible by 3.

$$
54: 3=18, \quad \text { bo } \quad 3 * 18=54
$$

Example $1.10 n=756$ is divisible by 3 because its sum of the digits $7+5+6=18$ is divisible by 3.

$$
756: 3=252 \quad \text { bo } \quad 3 * 254=756
$$

The divisibility test of $n=a_{1} a_{0}$ by 3 has a simple justification for two-digit numbers. A two-digit number has the tens digit $a_{1}$, and the second ones digit $a_{0}$.

$$
a_{1} a_{0}=a_{1} * 10+a_{0}=a_{1}(9+1)+a_{0}=9 * a_{1}+a_{1}+a_{0}
$$

The first component of $9 * a_{1}$ is divided by 3 bo

$$
9 * a_{1}: 3=3 a_{1}, \quad \text { or } \quad \frac{9 * a_{1}}{3}=3 a_{1}
$$

If the second component of the sum $a_{1}+a_{0}$ is divided by 3 , the sum is also divided by 3

Example 1.11 The number $n=57$ has tens digit 5 and units digit 7. So the sum of the digits $5+7=12$ divides 3 and the number 57 also divides 3 .
We do

$$
\begin{aligned}
& 57=5 * 10+7=5 *(9+1)+7=5 * 9+5+7 \\
& (5 * 9+5+7): 3=5 * 9: 3+12: 3=5 * 3+4=19
\end{aligned}
$$

- Numbers divisible by 4

$$
0.4,8,12,16,20,24,28,32,36,40,44,48,52, \ldots
$$

we write in general form

$$
n=4 k, \quad \text { for } \quad k=0,1,2,3, \ldots ;
$$

Clearly, the numbers of the form $4 * k, k=0,1,2,3, \ldots$; are divisible by 4 , since

$$
4 * k: 4=k, \quad \text { or } \quad \frac{4 * k}{4}=k
$$

for each natural $k=0,1,2,3,4, \ldots$;

- Numbers that are divisible by 5 .

$$
0.5,10,15,20,25,30,35,40,45, \ldots ;
$$

we write in general form

$$
n=5 k, \quad \text { for } \quad k=0,1,2,3, \ldots ;
$$

of course, the numbers of the form $5 * k, k=0,1,2,3, \ldots$; are divisible by 5 , because

$$
5 * k: 5=k, \quad \text { or } \quad \frac{5 * k}{5}=k
$$

for each natural $k=0,1,2,3,4, \ldots$;

The divisibility of $n$ by 5 can be found using the test:
The number $n$ is divisible by 5 if and only if its one digit is 0 or 5 .
Example $1.14 N=50$ is divisible by 5 because its one digit is 0 .

$$
50: 5=10, \quad \text { bo } \quad 5 * 10=50
$$

Example $1.15 N=265$ is divisible by 5 because its unit digit is 5 .

$$
265: 5=53, \quad \text { bo } \quad 5 * 53=265
$$

### 1.10 Dividing numbers by single digit numbers with remainder

Natural numbers that satisfy the divisibility test for 2 or 3 and 5 with the remainder 0 , we say they are divisible by 2 or 3 and 5 . However, there are many numbers that do not satisfy divisibility tests, then we execute dividion with remainder.

### 1.11 How to divide numbers with remainder

Consider the following examples
Example 1.12 Divide 13 by 3

$$
\frac{13}{3}=\overbrace{\frac{\overbrace{3 * 4+1}^{13}}{3}}^{13}=4+\frac{1}{3}
$$

The number 13 divided by 3 is 4 with the remainder 1 .
We divide the numbers according to the scheme

| 4 |  |
| ---: | :--- |
| - |  |
| 13 | $: 3$ |
| -12 |  |
| - |  |
| 1 |  |

Answer: The number 13 divided by 3 is 4 with the remainder of 1. We write the result of the division in the form:

$$
13=4 * 3+1
$$

Example 1.13 Divide 53 by 8

$$
\frac{53}{8}=\overbrace{\frac{\overbrace{6 * 8+5}}{83}}^{83}=6+\frac{5}{8}
$$

The number 53 is divided by 8 with the remainder 5 .
We divide the numbers according to the scheme

| 6 |  |
| ---: | ---: |
| - |  |
| 53 | $: 8$ |
| -48 |  |
| -- |  |
| 5 |  |

Answer: number 53 divided by 8 is 6 with the remainder of 5 We write the result of the division in the form:

$$
53=6 * 8+5
$$

Example 1.14 Divide the number 85 by 9

$$
\frac{85}{9}=\overbrace{\frac{9 * 9+4}{85}}^{9}=6+\frac{4}{9}
$$

85 is divided by 9 out of the remainder of 4 .
We divide the numbers according to the scheme

$$
\begin{array}{rr}
9 & \\
- & \\
85 & : 9 \\
-81 & \\
- & \\
4 &
\end{array}
$$

Answer: 85 over 9 is 9 with a remainder of 4
We write the result of the division in the form:

$$
85=9 * 9+4
$$

Generally, we write that the number $n$ is divided by the number $d$ with the remainder of $r$ according to the formula

Example 1.15 We divide $n$ by d

$$
\frac{n}{d}=\frac{\overbrace{k * d+r}^{n}}{d}=k+\frac{r}{d}
$$

The number $n$ divides by $d$ with the remainder in $r$.
We divide the numbers according to the scheme

$$
\begin{array}{rr}
k & \\
- & \\
n & : d \\
-k * d & \\
- & \\
r &
\end{array}
$$

Answer: $n$ divided by $d$ equals $k$ with the remainder in $r$ We write the result of the division in the form:

$$
n=k * d+r
$$

### 1.12 Dividing numbers by two-digit numbers with remainder

### 1.12.1 Examples

Example 1.16 Divide 78 by 42

$$
\frac{78}{42}=\overbrace{\frac{\overbrace{42+36}}{48}}^{42}=1+\frac{36}{42}
$$

78 is divided by $42 w i t h t h e r e m a i n d e r ~ 36$.
We divide the numbers according to the scheme

$$
\begin{array}{rr}
1 & \\
- & \\
78 & : 42 \\
-42 & \\
- & \\
36 &
\end{array}
$$

The answer: 78 divided by 42 is 1 , remainder 36 We write the result of the division in the form:

$$
78=1 * 42+36
$$

Example 1.17 Divide the number 1190 by 25

$$
\frac{1190}{25}=\overbrace{\frac{47 * 25+15}{1190}}^{25}=47+\frac{15}{25}
$$

1190 is divided by 25 with the remainder 15.
Now we divide according to the scheme

| 47 |  |
| :---: | :---: |
| - |  |
| 1190 | $: 25$ |
| -100 |  |
| - |  |
| 190 |  |
| -175 |  |
| - |  |
| 15 |  |

Answer: 1190 over 25 is 47 with a remainder of 15
We write the result of the division in the form:

$$
1190=47 * 25+15 \quad \text { or } \quad \frac{1190}{25}=47+\frac{15}{25}=47+\frac{3}{5}
$$

Example 1.18 Divide 1995 by 17

$$
\frac{1995}{17}=\overbrace{\frac{117 * 17+6}{17}}^{1995}=117+\frac{6}{17}
$$

1995 is divided by 17 withtheremainder 6 .

Now we divide according to the scheme

$$
\begin{array}{rl}
117 & \\
- & \\
1995 & : 17 \\
-17 & 17 \text { th ranks } 19 \text { th once, write } 1 \text { above dash } \\
- & 1 * 17=17 ; \text { subtract } 17 \\
29 & \\
-17 & \text { to difference } 2 \text { add next digit } 9 \\
- & \text { 17th ranks } 29 \text { th once, write second } 1 \text { above dash } \\
125 & \text { to difference } 12 \text { add next digit } 5 \\
-119 & 17 \text { th ranks } 125 \text { seven times } \\
- & 7 * 17=119 \text { write } 7 \text { above dash } \\
6 & \text { remainder } 6
\end{array}
$$

Answer: 1995 divided by 17 is 117 with a remainder of 6
We write the result of the division in the form:

$$
1995=117 * 17+6 \quad \text { or } \quad \frac{1995}{17}=117+\frac{6}{17}
$$

### 1.12.2 Questions

Question 1.16 Do a written hyphen

$$
(i) \quad 2546: 3, \quad(i i) \quad 5796: 9
$$

Question 1.17 Do a written hyphen

$$
\text { (i) } \quad 455: 13, \quad(i i) \quad 18011: 31
$$

Question 1.18 Do a written division of the remainder
(i) $2547: 3$,
(ii) $5766: 9$

Question 1.19 Prove that $\alpha_{3} \alpha_{2} \alpha_{1} \alpha_{0}$ is divisible by 3 if and only if the sum of the digits

$$
\alpha_{3}+\alpha_{2}+\alpha_{1}+\alpha_{0}
$$

is divisible by 3. Provide a necessary and sufficient condition that the number $\alpha_{3} \alpha_{2} \alpha_{1} \alpha_{0}$ is divisible by 9.

Question 1.20 Find the smallest integer greater than 2018, with the sum of its digits equal to 11 .

Question 1.21 Prove that the four-digit number $\alpha_{3} \alpha_{2} 25$ is divisible by 25 for any digits $\alpha_{3}, \alpha_{2}$.

## Question 1.22 .

(a) Write down the set of numbers divisible by 3 with a general formula. Write down 5 consecutive numbers divisible by 3.
(b) Write a general formula set of numbers divisible by 3 with remainder 1. Write 6 consecutive numbers divisible by 3 with remainder 1
(c) Write a general formula for the set of numbers divisible by 3 with remainder 2. Write the first 7 consecutive numbers divisible by 3 with a remainder 2.

Question 1.23 The brothers Antek, Bolek, Wacek and Sta received a total of PLN 400 from their father for school purchases. Bolek spent $4 z$ more than Antek, Wacek spent less than Bolek, Sta spent the same as Wacek.
How much did each of them spend on school purchases?
Question 1.24 There were cows, sheep, chickens and geese on the farm. Sheep were 2 times more than cows, geese were 4 times more than sheep, there were 6 times more chickens than geese. Together they had 124 legs. How many cows, sheep, chickens and geese were on the farm?

## Chapter 2

## Rational and irrational numbers

### 2.1 Fractions

The quotient of two integers $p$ and $q$

from the ab
is a fraction where $p$ is the numerator of a fraction and $q \neq 0$ different from zero is the denominator of the fraction.
For example, the fraction

has the numerator $p=5$ and the denominator $q=8$.
Note that the numerators of fractions

$$
\frac{1}{1}, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}, \quad \frac{1}{6}, \quad \frac{1}{7}, \quad \frac{1}{8}, \quad \frac{1}{9}
$$

are equal to 1 , but the denominators of these fractions are consecutive numbers $1,2,3,4,5,6,7,8,9$ However, the fractions given below have different numerators and denominators.

$$
\frac{1}{2}, \quad \frac{2}{3}, \quad \frac{3}{4}, \quad \frac{4}{5}, \quad \frac{5}{6}, \frac{9}{7}, \frac{11}{8}, \frac{10}{9}, \frac{11}{10}, \frac{12}{11}, \frac{13}{12}
$$

Among the fractions, we distinguish proper fractions and improper fractions.
Fractions are fractions

$$
\frac{\overbrace{p}^{\underbrace{\text { numerator }}_{\text {denominator }}}}{q}, \quad p<q \quad q \neq 0
$$

in which the numerator $p$ is less than the denominator $q$.
Similarly, Improper fractions are of the same form

$$
\frac{\overbrace{p}^{\overbrace{\text { denominator }}^{\text {numerator }}}}{\underbrace{\text { n. }}_{\text {qum }}}, \quad p>q \quad q \neq 0
$$

but their the numerator $p$ is greater than the denominator $q$.
We also write improper fractions in the form mixed numbers distinguishing the integer part For example, improper fraction

$$
\frac{5}{2}
$$

we write conventionally in the form of a mixed number $2 \frac{1}{2}$,

$$
\begin{gathered}
\frac{5}{2}=2+\frac{1}{2}=2 \frac{1}{2} \\
\frac{1}{2}, \quad \frac{2}{3}, \quad \frac{3}{4}, \quad \frac{4}{5}, \quad \frac{5}{6}, \quad \frac{9}{7}, \quad \frac{11}{8}, \quad \frac{10}{9}, \quad \frac{11}{10}, \quad \frac{12}{11}, \quad \frac{13}{12}
\end{gathered}
$$

Among the fractions, we distinguish proper fractions and improper fractions.
Proper fractions are fractions
in which the numerator $p$ is less than the denominator $q$.
Similarly, Improper fractions are fractions

$$
\frac{\overbrace{p}^{\overbrace{\text { denominator }}^{\text {numerator }}}, \quad p>q}{\underbrace{}_{\text {dem }}}, \quad q \neq 0
$$

in which the numerator $p$ is greater than the denominator $q$.
We also write improper fractions in the form mixed numbers. Denoting the integer part For example, improper fraction

$$
\frac{5}{2}
$$

we write conventionally in the form of a mixed number $2 \frac{1}{2}$,

$$
\frac{5}{2}=2+\frac{1}{2}=2 \frac{1}{2}
$$

Example 2.1 Replace with improper fraction
to a mixed number

Do the division by excluding all of this fraction

$$
9: 4=2+\frac{2}{4}=2+\frac{1}{2}
$$

From where the improper $\frac{9}{4}$ is equal to the number mix $2 \frac{1}{2}$, wewrite

$$
\frac{9}{4}=2 \frac{1}{2}
$$

Example 2.2 Replace the hash number

$$
1 \frac{3}{4}
$$

to an improper fraction
Given improper fraction we write it the sum of the whole mixed number 1 add the fraction $\frac{3}{4}$, as below

$$
1 \frac{3}{4}=1+\frac{3}{4}=\overbrace{\frac{4}{4}}^{1}+\frac{3}{4}=\overbrace{\frac{4+3}{4}}^{7}=\frac{7}{4}
$$

Why the mixed number $1 \frac{3}{4}$ is equal to the improper frac $\frac{7}{4}$, we write

$$
1 \frac{3}{4}=\frac{7}{4}
$$

### 2.2 Adding Fractions

Adding fractions with the same denominators. Fractions with the same denominators add the numerators, leaving the same denominator.

Example 2.3 Add fractions

$$
\begin{aligned}
& \frac{1}{2}+\frac{1}{2}=\frac{1+1}{2}=\frac{2}{2}=1 \\
& \frac{1}{3}+\frac{1}{3}+\frac{1}{3}=\frac{1+1+1}{3}=\frac{3}{3}=1 \\
& \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{1+1+1+1}{4}=\frac{4}{4}=1
\end{aligned}
$$

Example 2.4 Add fractions

$$
\begin{aligned}
& \frac{1}{2}+\frac{3}{2}=\frac{1+3}{2}=\frac{4}{2}=2 \\
& \frac{1}{3}+\frac{2}{3}+\frac{4}{3}=\frac{1+2+4}{3}=\frac{7}{3}=2 \frac{1}{3} \\
& \frac{1}{4}+\frac{2}{4}+\frac{3}{4}+\frac{5}{4}=\frac{1+2+3+5}{4}=\frac{11}{4}=2 \frac{3}{4}
\end{aligned}
$$

Adding singles with different denominators. To add fractions with different denominators, find a common denominator. This may be the lowest common multiple of the denominators.

Example 2.5 Add fractions

$$
\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{3+2}{6}=\frac{5}{6}
$$

common amultipleof $2 i 3$ equalto 6
$\frac{1}{3}+\frac{1}{4}+\frac{2}{5}=\frac{20}{6}+\frac{15}{60}+\frac{24}{60}=\frac{20+15+24}{60}=\frac{59}{60}$
common multiple 3, $4 i 5$ equalto 60
$\frac{1}{4}+\frac{3}{5}=\frac{5}{20}+\frac{12}{20}=\frac{5+12}{20}=\frac{17}{20}$
common amultipleof $4 i 5$ equalto 20
Zasanie 2.1 Replace with the fraction

$$
\frac{24}{5}
$$

to a mixed number
Zasanie 2.2 Replace the hash number

$$
5 \frac{4}{9}
$$

in the fraction

### 2.3 Subtraction of fractions

Subtracting fractions with the same denominators. Subtracting fractions with the same denominators as follows:
we subtract the numerators and leave the same denominator
Example 2.6 Subtract fractions

$$
\begin{aligned}
& \frac{1}{2}-\frac{1}{2}=\frac{1-1}{2}=0 \\
& \frac{2}{3}-\frac{1}{3}=\frac{2-1}{3}=\frac{1}{3} \\
& \frac{4}{5}-\frac{2}{5}-\frac{1}{5}=\frac{4-2-1}{5}=\frac{1}{5}
\end{aligned}
$$

Subtracting fractions with different denominators. By subtracting fractions with different denominators, find a common denominator. It may be the lowest common multiple of the denominators.

Example 2.7 Subtract fractions

$$
\begin{aligned}
& \frac{5}{9}-\frac{1}{3}=\frac{5-3 * 1}{9}=\frac{2}{9} \\
& \quad \text { common multiple; } 9 \text { i } 3 \text { equals } 9 \\
& \frac{33}{25}-\frac{21}{50}=\frac{2 * 33-21}{50}=\frac{45}{50}=\frac{9}{10} \\
& \quad \text { common multiple } 25 i 50 \text { rwna } 50 \\
& \frac{14}{15}-\frac{2}{5}+\frac{2}{3}=\frac{14-3 * 2+5 * 2}{15}=\frac{14-6+10}{15}=\frac{18}{15} \\
& \text { common multiple } 155 i 3 \text { rowna } 15 \\
& \frac{253}{500}-\frac{126}{1000}=\frac{2 * 253-126}{1000}=\frac{506-126}{1000}=\frac{380}{1000} \\
& \text { common multiple } 500 \text { i } 1000 \text { equalto } 1000
\end{aligned}
$$

Zasanie 2.3 Subtract fractions
(a) $\frac{5}{9}-\frac{2}{9}$
(b) $\frac{12}{5}-\frac{7}{5}$

Zasanie 2.4 Subtract fractions
(a) $\frac{15}{4}-\frac{3}{8}$
(b) $\frac{43}{6}-\frac{23}{7}$

### 2.4 Multiply fractions

The operation of multiplying fractions is very simple.
Fraction $\frac{p}{q}, q \neq 0$ multiplied by the fraction $\frac{s}{t}, \neq 0$ according to the scheme: numerator times numerator, denominator times denominator

$$
\frac{p}{q} * \frac{s}{t}=\frac{p * s}{q * t}, \quad q \neq 0, \quad t \neq 0
$$

Example 2.8 Multiply fractions
(a) $\frac{2}{3} * \frac{4}{5}=\frac{2 * 4}{3 * 5}=\frac{8}{15}$
(b) $\frac{2}{3} * \frac{(-4)}{5}=\frac{2 *(-4)}{3 * 5}=-\frac{8}{15}$
(c) $\frac{10}{13} * \frac{21}{25}=\frac{10 * 21}{13 * 25}=\frac{210}{273}$

Zasanie 2.5 Multiply fractions
(a) $\frac{5}{3} * \frac{4}{5}$
(b) $\frac{7}{9} * \frac{3}{5}$

### 2.5 Division of fractions

The operation for dividing fractions is very simple.
The fraction $\frac{p}{q}, q \neq 0$ divided by the fraction $\frac{s}{t}, s \neq 0$ according to the scheme: numerator times denominator, denominator times numerator

$$
\frac{p}{q}: \frac{s}{t}=\frac{p * t}{q * s}, \quad q, s \neq 0, \quad p, t \neq 0
$$

Example 2.9 Divide fractions
(a) $\frac{2}{3}: \frac{4}{5}=\frac{2}{3} * \frac{5}{4}=\frac{2 * 4}{3 * 5}=\frac{8}{15}$
(b) $\frac{-5}{9}: \frac{1}{3}=\frac{(-5) * 3}{9 * 3}=\frac{10}{12}$
(c) $\frac{10}{13}: \frac{21}{25}=\frac{10 * 25}{13 * 21}=\frac{250}{273}$

Zasanie 2.6 Divide fractions
(a) $\frac{5}{4}: \frac{4}{5}$
(b) $-\frac{15}{3}: \frac{18}{5}$
(c) $\frac{135}{4}:\left(-\frac{5}{48}\right)$

$$
\text { (d) }-\frac{48}{7}:\left(-\frac{16}{3}\right)
$$

Let us note that rational numbers also include all integers

$$
\ldots-5,-4,-3,-2-,-1,0,1,2,3,4,5, \ldots
$$

Generally, for integers $p$ and $q \neq 0$ fraction

$$
\frac{p}{q}
$$

is not an integer, if $q \neq 1$. For $q=1$, the fraction is an integer. The set of all integers together with the set of all possible fractions form the set of rational numbers. The set of rational numbers is marked with the letter $W$ and we write

$$
W=\left\{\frac{p}{q}: \text { for integers numbers } p \text { i } q \neq 0\right\}
$$

We can easily check that the set of rational numbers is closed because of the four arithmetic operations addition, subtraction, multiplication and division by numbers other than zero.

That is, for any rational numbers $w_{1}, w_{2} \in W$ the result of four operations is a rational number

$$
w_{1}+w_{2} \in W, \quad w_{1}-w_{2} \in W, \quad w_{1} * w_{2} \in W, \quad \frac{w_{1}}{w_{2}} \in W, \quad w_{2} \neq 0
$$

For example, for

$$
w_{1}=-\frac{2}{3} \in W, w_{2}=\frac{3}{4} \in W
$$

sum

$$
w_{1}+w_{2}=\frac{2}{3}+\frac{3}{4}=\frac{2 * 4+3 * 3}{12}=\frac{8+9}{12}=\frac{17}{12}=\in W .
$$

is a rational number
For

$$
w_{1}=-\frac{1}{2} \in W, w_{2}=\frac{2}{3} \in W
$$

difference

$$
w_{1}-w_{2}=\frac{1}{2}-\frac{2}{3}=\frac{1 * 3-2 * 3}{6}=\frac{3-6}{6}=-\frac{3}{6}=-\frac{1}{2} \in W
$$

is a rational number
For

$$
w_{1}=\frac{2}{3} \in W, w_{2}=\frac{3}{4} \in W
$$

product

$$
w_{1} * w_{2}=\frac{2}{3} * \frac{3}{4}=\frac{2 * 3}{3 * 4}=\frac{6}{12}=\frac{1}{2}=\in W .
$$

is a rational number
Also, for the numbers

$$
w_{1}=\frac{2}{3} \in W, w_{2}=\frac{3}{4} \in W
$$

quotient

$$
w_{1}: w_{2}=\frac{\frac{2}{3}}{\frac{3}{4}}=\frac{2 * 4}{3 * 3}=\frac{8}{9} \in W
$$

is a rational number
Let us observe that the set of rational numbers is dense everywhere. It means that, between two different rational numbers $w_{1}, w_{2}$ there are "many" other rational numbers, for example their arithmetic mean $\frac{w_{1}+w_{2}}{2} \in W$.
Moreover, the set of rational numbers $W$ is the smallest numeric set closed because of four arithmetic operations. Namely, let's assume for a moment that the rational number $x$ does not belong to the set $W,(x \notin W)$. Since each rational number has the form $\frac{p}{q}$ for some integers $p$ and $q \neq 0$. That is, there are no rational numbers outside of $W$.

Rational numbers are represented as points on the number line


Number line of rational numbers

### 2.6 Irrational numbers and real numbers

So far we have learned about the $N$ set of integers, the $C$ set of integers and the $W$ set of rational numbers. We know that in the set of natural numbers, two arithmetic operations are possible, addition and multiplication, while the result of subtracting or dividing two natural numbers may not be a natural number.
The set of integers $C$ is an extension of the set of natural numbers $N$. So all natural numbers belong to the set of integers, we write $N \subset C$. In the set $C$ integers, three arithmetic operations add subtraction and multiplication are possible, and the result of dividing two integers may not be an integer.
The set of $C$ rational numbers is extended by $W$. So all the integers belong to the set $W$ of rational numbers, we write $C \subset W$. In the set $W$ of rational numbers all four arithmetic operations add, subtract and multiply and divide.
Note that in the set of rational numbers $W$ the inverse of exponentiation is not always feasible.
For example, there is no rational number $x$ whose squared is 2 . Another equation

$$
x^{2}=2
$$

there is no solution in the set of rational numbers.
Indeed, if there were a rational number

$$
x=\frac{p}{q}, q \neq 0
$$

with the greatest common divisor $G C D(p, q)=1$ then this rational number would solve the equation

$$
\left(\frac{p}{q}\right)^{2}=2, \quad \text { and } \quad p^{2}=2 q^{2}
$$

Then the integer $p$ would be an even number, that is, $p=2 k$ for some integer $k$. In this case, $q$ would also have to be an even number, that is

$$
q=2 s
$$

for some complete $s$.
As a consequence, we have the inequality $G C D(p, q)>=2$, which denies the existence of a rational number in the form of an irreducible frac $\frac{p}{q}$, where the greatest common divisor of the numerator $p$ i denominator $q, G C D(p, q)=1$.
Another extension of the sets of numbers

$$
N, C, W
$$

is the set of real numbers $R$ in which the inverse exponentiation is doable.
The set of real numbers includes all rational numbers and all irrational numbers such as

$$
\sqrt{2}, \quad \sqrt[3]{5}, \quad \sqrt[5]{7}, \quad \pi, \ldots
$$

$\sqrt{x^{2}}=|x|$, therefore $\sqrt{4}=2$, but -2 is not the square root of 4


The set of real numbers is denoted by the letter $R$, wewrite

$$
R=\{\ldots \ldots .-\pi,-3,-\sqrt{5},-2,-\sqrt{2}-1,0,1, \sqrt{2}, 2, \sqrt[3]{9}, 3, \pi \ldots ;\}
$$

### 2.6.1 Questions

Question 2.1 Calculate the value of an arithmetic expression

$$
20 \frac{\left(\frac{1}{2}+\frac{1}{3}\right)\left(\frac{2}{3}-\frac{1}{2}\right)}{\left(2-\frac{1}{3}\right)\left(1+\frac{2}{3}\right)}
$$

Question 2.2 Calculate the value of an arithmetic expression

$$
5\left[\left(\frac{4}{5}+\frac{7}{10}\right)\left(\frac{1}{5}+\frac{1}{10}\right)+\left(\frac{4}{5}-\frac{7}{10}\right)\left(\frac{4}{5}+\frac{7}{10}\right)\right]
$$

Question 2.3 Calculate the value of an arithmetic expression

$$
\frac{1}{2}\left(\frac{2}{5}+1.5\right):\left(\frac{5}{7}-1 \frac{2}{3}\right)
$$

Question 2.4 Calculate the value of an algebraic expression

$$
36\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{a}{b}-\frac{b}{a}\right)
$$

for $a=3$ and $b=2$
Question 2.5 Perform arithmetic operations

$$
a * b, \quad a-b, \quad b: a
$$

for $a=3+\sqrt{7}, \quad b=4-2 \sqrt{7}$
Question 2.6 Calculate the value of the expression

$$
\sqrt{67-\sqrt[3]{27}}
$$

Question 2.7 Prove that $\sqrt{3}$ is an irrational number
Question 2.8 Prove that $\sqrt[3]{7}$ is an irrational number
Question 2.9 Find the values of the $a$ and $b$ parameters for which

$$
a \sqrt{b}=\sqrt{50}+\sqrt{128}+\sqrt{162}
$$

Question 2.10 For the harvest

$$
A=\{x:-\infty<x<5\} \quad \text { and } \quad B=\{x: 2<x \leq 9\}
$$

Mark on the number line the alternative sets $A \smile B i$ and their conjunction $A \frown B$.

### 2.7 Absolute value

Absolute value of a number $x$ is the distance of the point $x$ from the origin 0 . Therefore the absolute value of $x$ is always non-negative.
The absolute value of $x$ is defined as follows

## Definicja 2.1

$$
|x|=\left\{\begin{aligned}
& x, \text { when } \\
&-x \text { when } \\
&-x<0
\end{aligned}\right.
$$

For example, $|5|=5$ because $5>0$, and $|-5|=-(-5)=5$, if $x=-5<0$. Also the absolute value of $x$ is given by the formula

$$
|x|=\sqrt{x^{2}}
$$

Absolute value plot of real numbers $|x|$


Plot of absolute value $y=|x|$

Let us observe that absolute values $|x|$, satisfy the following inequality

$$
|x| \leq a, \quad \text { if and only if } \quad-a \leq x \leq a, ; \quad a \geq 0
$$

Indeed, we note that

$$
|x| \leq a, \quad \text { if } \quad x \leq a \quad \text { i } \quad-x \leq a, \quad \text { this means } \quad-a \leq x \leq a
$$

Let's mark values $x$ that satisfy $-a \leq x \leq a$.


Similarly, the interval $[a, b]$ beginning at $a$ and ending at $b$, that is the set of points $x$ lying between the points $a$ and $b$ as set

$$
[a, b]=\{x \in R: a \leq x \leq b\}
$$

The length of the interval $[a, b]$, that is the distance of the point $a$ from the point $b$, is equal to the absolute value of the difference $|b-a|$.

Example 2.1 Solve the equation

$$
|2 x-3|=5
$$

Mark the solution on the number line.
Solution. From the absolute value definition

$$
|2 x-3|=\left\{\begin{aligned}
2 x-3=5, & \text { when } \quad 2 x-3 \geq 0, \\
-(2 x-3)=5 & \text { when } \quad x=4 \\
-2 x+3 \leq 0, & \text { to } \quad x=-1
\end{aligned}\right.
$$

Letus see the solution $\mathrm{x}=-1$ and $x=4$ below on the number line.


Solution: $x=-1$ or $x=4$.

Example 2.2 Solve inequalities

$$
|x-3| \leq 2
$$

Solution. By definition, absolute value inequality

$$
|x-3| \leq 2
$$

it is equivalent to double inequality

$$
-2 \leq x-3 \leq 2, \quad \text { or } \quad 1 \leq x \leq 5
$$

Answer: $1 \leq x \leq 5$.
Example 2.3 Find the set of points that satisfy the inequality

$$
|x-1|+|x+1| \leq 1
$$

Select this set on the number line.
Solution. From the definition of the absolute value we find

1. for $\quad x+1 \leq 0, \quad|x+1|=-(x+1)=-1-x$, and $|x-1|=-(x-1)=-1-x$
$|x-1|+|x+1|=1-x-1-x=-2 x \leq 1$,
when $x \geq-\frac{1}{2}, \quad$ then inequality no has solutions
2. for $\quad-1 \leq x \leq 1, \quad|x-1|=(x-1)=x-1$, and $|x+1|=-(x+1)=-1-x$
$|x-1|+|x+1|=x-1-1-x=-2 \leq 1$,
when $-1 \leq x \leq 1$, then inequality is true for $-1 \leq x \leq 1$
3. for $\quad x-1 \geq 0, \quad|x-1|=x-1$, and $|x+1|=x+1$

$$
|x-1|+|x+1|=(x-1)+(x+1)=2 x \leq 1
$$

when $x \leq \frac{1}{2}, \quad \quad$ then inequality no has solutions

Answer: The inequality

$$
-1 \leq x \leq 1
$$

is true for for all values $|x| \leq 1$.
Let us note that the distance of each point $x \in[-1,1]$ from the point -1 plus the distance of this point $x$ from the point 1 is equal to 1 . Thus, the inequality is also true for $x=-1$ or $x=1$. Then the equal sign holds.
Let us mark the solution in the number line


### 2.7.1 Questions

Question 2.11 Solve the equation

$$
|3 x-5|=4
$$

Mark the solution on the number line.
Question 2.12 Solve the equation

$$
|2 x-3|=5
$$

Mark the solution on the number line.
Question 2.13 Solve the inequality

$$
|x-5| \leq 2
$$

Mark the solution on the number line.
Question 2.14 Give the set of points that satisfy the inequality

$$
|x|+|x-2| \leq 2
$$

Select this set on the number line

## Chapter 3

## Arithmetic and algebraic expressions

Let's start with framing the concepts of arithmetic and algebraic expressions.
Definicja 3.1 A sequence of numbers connected by four arithmetic operations: addition, subtraction, multiplication, and division by numbers different from zero is called algbraic expressions

For example, the expression

$$
\frac{3 * 4+6: 2-2 * 3}{2^{3}+3^{2}-8: 2}
$$

is the arithmetic expression consisting of a sequence of numbers

$$
\begin{array}{ll}
\text { numerator }: & 3,4,6,2,2,3, \\
\text { denominator }: & 2,2,2,3,3,2,8,2
\end{array}
$$

combined with arithmetic operations

$$
*,+,:,-, *, /,,+, \cdots-, ;:
$$

We define algebraic expressions similarly. Namely
Definicja 3.2 A sequence of numbers or letters connected by four arithmetic operations: addition, subtraction, multiplication, and division by numbers or letters is called algbraic expressions

For example

$$
\frac{a * 4+x: 2-2 * 3}{x^{3}+3^{2}-b: 2}
$$

is the algebraic expression consisting of a sequence of numbers and letters

$$
a, 4, x, 2,2,3, x, 3,3,3,2, b, 2
$$

with arithmetic operations $*,+,:,-, *, /,,+,,-, ;$ Their take form of arithemetic exsspretion for $a=3, x=6, b=8$.

To evaluate an arithmetic expression, we execute operations in the followig order first we perform multiplication and division operations.
second we perform additiona and subtruction operations.

The order of execution of arithmetic operations in an arithemetic expression can be changed by parentheses

### 3.1 Questions

Question 3.1 Calculate the value of an arithmetic expression
(a) $12+14+24$
(b) $50-24-8$

Question 3.2 Calculate the value of an arithmetic expression respecting the order of operations
(a) $18-16+2 * 8$
(b) $5 * 6+24: 3$

Question 3.3 Calculate the value of an arithmetic expression with parentheses

$$
\begin{aligned}
& \text { (and) } 3 *(4+6)-2 *(3+5) \\
& \text { (b) }(50-40) * 2-(10+6): 2
\end{aligned}
$$

Question 3.4 Calculate the value of an arithmetic expression

$$
5^{2} * 2^{3}+3^{2} * 2^{3}-4^{2} * 5^{2}
$$

Question 3.5 Calculate the value of the artmetic expression

$$
\frac{3^{3} * 2^{3}-3^{2} * 2^{2}}{3 * 2^{3}+2 * 3}
$$

Re: 6
Question 3.6 Calculate the value of an arithmetic expression

$$
\frac{\frac{2}{5} * \frac{3}{5}-\frac{2}{9} * \frac{3}{8}}{\frac{5}{3} * \frac{2}{5}+\frac{3}{7} * \frac{7}{3}}
$$

Question 3.7 Calculate the value of an arithmetic expression

$$
\frac{\frac{1}{3} * \frac{2}{5}-\frac{1}{2} * \frac{3}{5}}{\frac{2}{3} * \frac{1}{4}+\frac{3}{4} * \frac{4}{3}}
$$

Question 3.8 Calculate the value of an arithmetic expression with parentheses

$$
\frac{\left(\frac{1}{3}-\frac{1}{2}\right)\left(\frac{2}{3}-\frac{1}{2}\right)}{\left(\frac{2}{3}+\frac{3}{4}\right)\left(\frac{4}{3}+\frac{5}{4}\right)}
$$

Question 3.9 Calculate the value of an arithmetic expression with parentheses

$$
\frac{\left(\frac{2}{5} * \frac{3}{5}-\frac{2}{9} * \frac{3}{8}\right)\left(\frac{3}{5} * \frac{3}{5}-\frac{1}{5} * \frac{3}{5}\right)}{\left(\frac{5}{3} * \frac{2}{5}+\frac{3}{7} * \frac{7}{3}\right)\left(\frac{5}{3} * \frac{2}{5}+\frac{3}{7} * \frac{7}{3}\right)}
$$

Question 3.10 Calculate the value of an arithmetic expression with parentheses

$$
\frac{\left(\frac{1}{3} * \frac{2}{5}-\frac{1}{2} * \frac{3}{5}\right)\left(\frac{1}{3} * \frac{2}{5}-\frac{1}{2} * \frac{3}{5}\right)}{\left(\frac{2}{3} * \frac{1}{4}+\frac{3}{4} * \frac{4}{3}\right)\left(\frac{2}{3} * \frac{1}{4}+\frac{3}{4} * \frac{4}{3}\right)}
$$

### 3.2 Algebraic expressions

${ }^{1}$ Remember that apart from arithmetic expressions, we have algebraic expressions. In algebraic expressions we allow letters, symbols with variable value. Thus, an algebraic expression is a sequence of numbers and letters joined by the arithmetic operations of addition, subtraction, multiplication, and division.

Example 3.1 Simplify the expression

$$
\frac{a^{2}-a}{a-1}-(a+1), \quad a>1
$$

Solution. Taking arithmetic operations, let's calculate

$$
\begin{aligned}
\frac{a^{2}-a}{a-1}-(a+1) & =\frac{\left(a^{2}-a\right)-(a-1)(a+1)}{a-1} \\
& =\frac{\left(a^{2}-a\right)-[a(a+1)-1(a+1)]}{a-1} \\
& =\frac{a^{2}-a-\left[a^{2}+a-a-1\right]}{a-1} \\
& =\frac{\left.a^{2}-a-a^{2}+1\right]}{a-1} \\
& =\frac{1-a}{a-1}=-\frac{1-a}{1-a}=-1 .
\end{aligned}
$$

### 3.2.1 Questions

Question 3.11 Calculate the value of the algebraic expression for the value $a=2$

$$
\frac{\frac{c}{3} * \frac{c}{5}-\frac{c}{3} * \frac{3}{c}}{\frac{c}{3} * \frac{1}{c}+\frac{3}{c} * \frac{c}{3}}
$$

Question 3.12 Calculate the value of the algebraic expression for the value $b=1$

$$
\frac{\frac{2}{b} * \frac{3}{b}-\frac{2}{b} * \frac{3}{b}}{\frac{b}{3} * \frac{b}{5}+\frac{b}{7} * \frac{b}{3}}
$$

[^0]Question 3.13 Calculate the value of ść of the algebraic expression for the value $c=3$

$$
\frac{\frac{c}{3} * \frac{c}{5}-\frac{c}{3} * \frac{3}{c}}{\frac{c}{3} * \frac{1}{c}+\frac{3}{c} * \frac{c}{3}}
$$

Question 3.14 Calculate the value of the algebraic expression for $a=2$

$$
\frac{\left(\frac{a}{3}-\frac{a}{2}\right)\left(\frac{2}{3}-\frac{a}{2}\right)}{\left(\frac{2}{3}+\frac{a}{4}\right)\left(\frac{4}{a}+\frac{a}{4}\right)}
$$

Question 3.15 Calculate the value of śc of the algebraic expression for $b=3$

$$
\frac{\left(\frac{b}{5} * \frac{3}{b}-\frac{b}{9} * \frac{3}{b}\right)\left(\frac{3}{b} * \frac{3}{b}-\frac{1}{b} * \frac{3}{5}\right)}{\left(\frac{b}{3} * \frac{2}{b}+\frac{3}{7} * \frac{b}{3}\right)\left(\frac{b}{3} * \frac{b}{5}+\frac{b}{7} * \frac{b}{3}\right)}
$$

Question 3.16 Find the value of the algebraic expression for $c=1$

$$
\frac{\left(\frac{b}{5} * \frac{3}{b}-\frac{b}{9} * \frac{3}{b}\right)\left(\frac{3}{b} * \frac{3}{b}-\frac{1}{b} * \frac{3}{5}\right)}{\left(\frac{b}{3} * \frac{2}{b}+\frac{3}{7} * \frac{b}{3}\right)\left(\frac{b}{3} * \frac{b}{5}+\frac{b}{7} * \frac{b}{3}\right)}
$$

### 3.3 Linear algebraic expressions

Algebraic expression of the form

$$
a * x+b
$$

is called linear due to the variable $x$, and given coefficients $a$ and $b$.
For example

$$
\begin{array}{llll}
2 * x+1, & \text { where coefficients } & a=2, & b=1 \\
-5 * x+4, & \text { where coefficients } & a=-5, & b=4
\end{array}
$$

### 3.3.1 Questions

Question 3.17 Write a linear algebraic expression with coefficients
(i) $\quad a=5, \quad b=-25$
(ii) $\quad a=\frac{3}{5}, \quad b=\frac{2}{9}$
(iii) $\quad a=-\frac{13}{15}, \quad b=-\frac{15}{29}$

### 3.4 Linear equations

Equation in the form

$$
a * x+b=0
$$

or any other equation that can be simpkified into this form is called a linear equation with respect to unknown $x$ and given coefficients $a$ and $b$.
The solution of a linear equation with the unknown $x$ is each number when is substituted for $x$ satisfies this equation.

Example 3.2 Consider the linear equation

$$
\begin{array}{rrrrrr}
2 * x-4=0, & x=2, & \text { since } & 2 * 2-4=0, & \text { for } & a=2, \\
-3 * x+3=0, & x=1, & \text { since } & -3 * 1+3=0, & \text { for } & a=-3,
\end{array} \quad b=3
$$

Example 3.3 Solve linear equation

$$
2 x-1=0, \quad a=2, \quad b=-1
$$

## Solution.

We transfer the number -1 to the right side, changing the sign to the opposite and divide both sides of this equation by 2

$$
2 x=1 \mid: 2
$$

This is how we find a solution

$$
x=\frac{1}{2}
$$

Substituting $x=\frac{1}{2}$ to the equation, we check that the obtained solution satisfies this equation.
Namely for $x=\frac{1}{2}$, we have

$$
2 x-1=2 \frac{1}{2}-1=1-1=0
$$

We see that the solution $x=\frac{1}{2}$ satisfies this equation.

Example 3.4 Solve linear equation

$$
5 x+3=3 x+5
$$

## Solution.

We write the number -3 on the right side and $-3 x$ on the left side

$$
5 x-3 x=5-3 \quad \text { or } \quad 2 x=2
$$

We find the solution

$$
x=1
$$

Substituting $x=1$ to the equation, we check that the obtained solution satisfies this equation.
Namely for

$$
5 * 1+3=3 * 1+5 \quad 8=8
$$

We see that the solution $x=1$ satisfies this equation.
General scheme for solving a linear equation.

$$
\begin{aligned}
& a x+b=0, \\
& a \neq 0, \\
& a x=-b, \\
& x=-\frac{b}{a},
\end{aligned}
$$

Question 3.18 Solve the equation
(i) $3 x-12=0$
(ii) $5 x+20=10$
(iii) $\frac{3}{4} x+\frac{5}{8}=1$
(iv) $\quad 1.5 x+2.9=7$

Question 3.19 The little shepherd saw flying storks and shouted, I think, there are 100 of them. The older shepherd replied much less, if there were twice as many, and a half as many, and a quarter as many, and if you were to fly with them, then 100 of them would be with you. How many storks were flying. in the sky?


Józef Chetmoński (1849-1914). Storks
Question 3.20 Franek read a book of 25 pages a day. He read the entire book within 3 days.
Calculate how many pages does this book have?

Question 3.21 Marysia bought 3 notebooks of 7 zlotys each. Kazik bought a ball for PLN 10 and a watch for PLN 35.

How much did Marysia pay for 3 notebooks?

How much did Kazik pay for the ball and the watch?
O How much more did Kazik pay for purchases from Marysia?
Bolek is 2 times older than Stefka, who is 7 years old. Olek is as old as Bolek and Stefka together.
(a) How old is Bolek?
(b) How old is Olek?

Question 3.22 There were crows on several trees. Janek said to his father
Dad,
I see a lot of crows in trees, I think there are 100 of them.

Father answered John, if it was twice as much and a half as much, then it would be 100 crows.

How many crows were sitting in the trees?
Question 3.23 Simplify the algebraic expression

$$
\frac{a^{2}-a}{a-1}-(a+1), \quad a>1
$$

### 3.5 Inequalities

Let us see on the number line values of the variable $x$ greater than zero.


Acute unequal $x>0$ of positive values of the variable $x$
Let us seethe number line those values of $x$ less than zero.


Acute unequal $x<0$ of negative values of the variable $x$
Let us see on the number line values of the variable $x$, that lie between the number 1 and the number 2 .

$x$ variable values between 1 and 2
Let us see on the number line values of the variable $x$, which lie between the number -2 and the number -1 or the number 1 and the number 2 .


The value of the variable $x$ in between -2 i -1 or 1 i 2
when $\quad-2 \leq x \leq-1 \quad$ or $\quad 1 \leq x \leq 2$

### 3.5.1 Examples and questions

Example 3.1 Solve the inequality

$$
3(x-1)<2(x+1)
$$

Mark on the number line those values of the variable $x$, for which the inequality is true.
Solution
We do multiplication on the left side by 3 and right side by 2 of the inequality to drop brackets. Then we get inequality

$$
3 x-3<2 x+2
$$

Always, we transfer the $x$ variable to the left side of the inequality with the opposite sign, and the numbers to the right side of the inequality also with the opposite sign to find solution

$$
3 x-2 x<2+3, \quad \text { solution } \quad x<5
$$

Let's mark the solution $x<5$ on number line


Acute inequality $x$ less than 5

Question 3.24 Solve the inequality

$$
\begin{aligned}
& \text { (i) } \quad 2 x-1>1 \\
& \text { (ii) } \quad 4 x-6 \leq 10
\end{aligned}
$$

Mark on the number line those values of the variable $x$, for which the inequalities ( $a$ ) and (b) $s$ are true.

Question 3.25 Solve the inequality
(i) $3(3 x-1)-2(2 x+1)<4(x-1)$
(ii) $3(x-2)+4(x+2) \leq 2 x+10$
(iii) $\quad \frac{x-1}{x+1}<1 \quad$ for $x \neq-1$

Mark on the number line those values of the variable $x$, for which the inequalities (i), (ii) and (iii) are true.

### 3.6 Decimal Fractions

Fractions with denominators $10,100,1000, \ldots$ we write in the form of decimals Every fraction can be cnverted to decimal.

$$
\frac{1}{10}=0.1, \quad \frac{1}{100}=0.01, \quad \frac{1}{1000}=0.001
$$

or

$$
\begin{aligned}
\frac{3}{10} & =0.3, \quad \frac{5}{100}, \quad 0.05 \\
\frac{35}{1000} & =0.035, \quad \frac{735}{1000}=0.735 \\
2 \frac{3}{10} & =2.3, \quad 10 \frac{12}{100}=10.12
\end{aligned}
$$

We have reciprocal relations, we convert decimal fractions to ordinary fractions

$$
\begin{aligned}
& 0.1=\frac{1}{10}, \quad 0.01=\frac{1}{100}, \\
& 0.001=\frac{1}{1000}, \quad 0.3=\frac{3}{10}, \\
& 0.05=\frac{5}{100}, \quad 0.035=\frac{35}{1000}, \\
& 0.735=\frac{735}{1000}, \quad 2.3=\frac{3}{10}, \\
& 10.12=10 \frac{12}{100} .
\end{aligned}
$$

We can convert every fraction into a decimal.
The first simple way to convert a decimal fraction is to write the fraction next to the denominator, $10,100,1000, \ldots$ This way is simple only for selected fractions.

### 3.6.1 Examples and questions

## Example 3.2

$$
\begin{aligned}
& \frac{1}{2}=\frac{1 * 5}{2 * 5}=\frac{5}{10}=0.1 \\
& \frac{3}{4}=\frac{3 * 25}{4 * 25}=\frac{75}{100}=0.25 \\
& \frac{7}{5}=\frac{7 * 20}{5 * 20}=\frac{140}{100}=1.4 \\
& \frac{15}{250}=\frac{15 * 4}{250 * 4}=\frac{60}{1000}=0.06
\end{aligned}
$$

The second way to convert fractions to decimals is to divide the numerator by the denominator.

Example 3.3 Convert $\frac{1}{4}$ to a decimal.
Solution. Divide $1=1.00$ by 4. Note that the zeros after the point do not change the value of 1 .

So we do the conversion following the scheme

$$
-\quad 0
$$

$$
\begin{array}{ll}
0,25 & \\
-- \\
1,00 & : 4 \\
0 & \\
-- & \\
10 \\
-8 & \\
-- & \\
20 & \\
-2 & -- \\
0 &
\end{array}
$$

Answer $\frac{1}{4}=0,25$
Question 3.26 Convert fraction to decimal
(i) $\frac{3}{5}$
(ii) $\frac{37}{50}$
(iii) $\frac{253}{250}$

Question 3.27 Convert fraction to decimal
(i) $\frac{2}{15}$
(ii) $\frac{23}{45}$
(iii) $\frac{37}{150}$

## Chapter 4

## Percentages and per mille

### 4.1 Persentage

$p \%$ percent is a fraction of $\frac{p}{100}$ with the numerator of $p$ and the denominator of 100.
For example
$1 \%$ one percent is a fraction of $\frac{1}{100}=0.01$ with the numerator of 1 and the denominator of 100.
$25 \%$ is a fraction of $\frac{25}{100}=0.25$ with the numerator of 25 and the denominator of 100 .
$100 \%$ to całość $\frac{100}{100}=1$ with the numerator 100 and the denominator 100
We calculate $p \%$ percent from the value $a$

$$
p \% * a=\frac{p}{100} * a
$$

as a fraction of the numerator $p$ and the denominator of 100 of $a$.

### 4.1.1 Examples

Example 4.1 Calculate $15 \%$ from the value of $a=60$

$$
15 \% * 60=\frac{15}{100} * 60=\frac{15 * 60}{100}=\frac{15 * 6}{10}=\frac{90}{10}=9
$$

Example 4.2 Calculate $25 \%$ from the value $a=3000$

$$
25 \% * 3000=\frac{25}{100} * 3000=\frac{25 * 3000}{100}=\frac{75000}{100}=750
$$

Conversely, if $p \% * a$ percent of the value of $a$, we calculate the value of $a$

Example $4.330 \%$ percent of a equals 600. Calculate the value of a

## Solution.

2ex]

$$
30 \% * a=600, \quad \frac{30}{100} * a=600, \quad a=\frac{600}{\frac{30}{100}}=\frac{600 * 100}{30}=2000
$$

### 4.1.2 Questions

Question 4.1 Calculate $75 \%$ from $a=2000$
Question 4.2 Calculate $15 \%$ from the value 'sci $a=4000$
Conversely, having $p \% * a$ a percentage of the value of $a$, calculate the value of $a$ given in the following exercises

Question $4.350 \%$ percent of the value of a equals 800 . Calculate the value of a

Question $4.430 \%$ percent of the value of $a$ is equal to 5000. Calculate the value of $a$

Question 4.5 The price of a square meter of fabric for window curtains was 50 from. First, the price was increased by $30 \%$ then decreased by $10 \%$ per square meter. How much did the customer pay for $10 \mathrm{~m}^{2}$ of material ?

Question 4.6 The price of the material together with 7\%VATwas 107 USD. Material VAT tax rose to $22 \%$. How much was the material with all VAT? How many percent did the price of the material increase?

### 4.2 Per mile

Promiles are fractions with a denominator of 1000 .
$p \% \%$ per milli is a fraction of $\frac{p}{1000}$ with the numerator of $p$ and the denominator of 1000 .

### 4.2.1 Examples

Example $4.41 \% \%$ one percent is a fraction of $\frac{1}{1000}=0.001$ the numerator 1 and the denominator 1000.
$25 \% \%$ is a fraction of $\frac{25}{1000}=0.025$ with the numerator of 25 and the denominator of 1000.
$1000 \% \%$ is the total of $\frac{1000}{1000}=1$ with the numerator 1000 and the denominator 1000.
We calculate $p \% \%$ percent from a

$$
p \% \% * a=\frac{p}{1000} * a
$$

as a fraction with the denominator of 1000 from a.

### 4.2.2 Questions

Question 4.7 Calculate $15 \% \%$ from the value of $a=3000$

$$
15 \% \% * 3000=\frac{15}{1000} * 3000=\frac{15 * 3000}{1000}=45
$$

Question 4.8 Calculate 25\%\% from the value of $a=3000$

$$
25 \% \% * 3000=\frac{25}{1000} * 3000=\frac{25 * 3000}{1000}=75
$$

Conversely, having $p \% \% * a$ a percentage of the value of $a$, we calculate the value of $a$

Example $4.530 \% \%$ percent of the value of a equals 600 . Calculate the value of $a$

## Solution.

$$
30 \% \% * a=600, \quad \frac{30}{1000} * a=600, \quad a=\frac{600}{\frac{30}{1000}}=\frac{600 * 1000}{30}=20.000
$$

Question 4.9 Calculate 75\%\% from the value of $a=2000$
Question 4.10 Calculate 15\%\% from the value $a=4000$
Conversely, having $p \% \% * a$ per mille of $a$, calculate the value of $a$ given below in exercises

Question $4.1150 \% \%$ per mille of $a$ is equal to 800. Calculate the value of $a$

Question $4.1230 \% \%$ per mille of $a$ is equal to 5000. Calculate the value of $a$.

### 4.3 Compound percentages

Let us introduce the following notations

- $K_{0}$ - initial capital
- $K_{n}$ - capital after $n$ years
- $p$ - annual interest rate
- n - number of years of savings

After the first year of saving, the capital $K_{0}$ will increase by $p \%$

$$
K_{1}=K_{0}+K_{0} \frac{p}{100}=K_{0}\left(1+\frac{p}{100}\right)
$$

After the second year saving 's your capital $K_{1}$ will increase by $p \%$

$$
K_{2}=K_{1}+K_{1} \frac{p}{100}=K_{1}\left(1+\frac{p}{100}\right)=K_{0}\left(1+\frac{p}{100}\right)^{2}
$$

Generally, using the principle of complete induction, if after $n-1$ years of saving, capital increases by $p \%$

$$
K_{n-1}=K_{0}\left(1+\frac{p}{100}\right)^{n-1}
$$

that after $n$ years saves

$$
K_{n}=K_{n-1}+K_{n-1} \frac{p}{100}=K_{0}\left(1+\frac{p}{100}\right)^{n}
$$

In this way, we got the formula for the final capital after $n$ years of saving

$$
K_{n}=K_{0}\left(1+\frac{p}{100}\right)^{n}
$$

### 4.3.1 Examples

Example 4.1 Calculate how much capital 150.000 will increase after 10 years, if the interest rate $p=5 \%$.

Solution. Using the formula, we calculate

$$
K_{10}=150.000\left(1+\frac{5}{100}\right)^{10}=150.000 * 1.05^{10}=150.000 * 1.62889=P L N 244,334
$$

Answer: Capital 150.000 will increase by 94334 PLN for 10 years if the annual interest rate is $p=5 \%$.

Repayment of the loan. Similarly, we calculate the compound interest on the loan. After the first year of repayment, the capital $K_{0}$ will decrease by $p \%$

$$
K_{1}=K_{0}-K_{0} \frac{p}{100}=K_{0}\left(1-\frac{p}{100}\right)
$$

After the second year of repayment, the capital $K_{1}$ will decrease by $p \%$

$$
K_{2}=K_{1}-K_{1} \frac{p}{100}=K_{1}\left(1-\frac{p}{100}\right)=K_{0}\left(1-\frac{p}{100}\right)^{2}
$$

In general, using the principle of complete induction, if after $n-1$ years of repayment, the capital decreases to the sum

$$
K_{n-1}=K_{0}\left(1-\frac{p}{100}\right)^{n-1}
$$

then, after $n$ years of repayment, the capital will decrease to the sum

$$
K_{n}=K_{n-1}-K_{n-1} \frac{p}{100}=K_{0}\left(1-\frac{p}{100}\right)^{n}
$$

Thus, we obtained the formula for the final capital lafter $n$ years of paying off the loan.

$$
K_{n}=K_{0}\left(1-\frac{p}{100}\right)^{n}
$$

Example 4.2 Calculate how much credit will decrease on capital PLN150.000 after 10 years of repayment. and after 150 years it will pay off if the interest rate $p=5 \%$.

Solution. Using the formula r, we calculate

$$
\begin{aligned}
K_{10} & =150.000\left(1-\frac{5}{100}\right)^{10}=150.000 * 0.95^{10}=150.000 * 0.598737=P L N 89,810 \\
K_{150} & =150.000\left(1-\frac{5}{100}\right)^{150}=150.000 * 0.95^{150}=150.000 * 0.0004555=P L N 68.33
\end{aligned}
$$

Answer: After 10 years, the loan will decrease by 60, 189.5PLN.However, after 150 yearstheloanwilldecreaseby 149,931.67 PLN.

## Chapter 5

## Arithmetic and geometric sequences and series

Arithmetic and geometric sequences and series are an important part of primary and secondary school curricula.
In this chapter the properties of arithmetic and geometric sequences and series are described and supported by exercises.

### 5.1 Arithmetic sequences and series

Let us consider an arithmetic sequence of numbers with terms

$$
a_{0}, \quad a_{0}+r, \quad a_{0}+2 r, \ldots, a_{0}+n * r \quad n=0,1,2, \ldots ;
$$

where $a_{0}$ is the first term of the sequence and the difference

$$
r=a_{k+1}-a_{k} \quad k=0,1,2,3, \ldots n-1
$$

The general term of the sequence $a_{n}$ is of the form

$$
a_{n}=a_{0}+n * r \quad n=0,1,2, \ldots ;
$$

For example, a sequence of consecutive natural numbers

$$
0,1,2, \ldots ;
$$

is an arithmetic sequence with the first term $a_{0}=0$, difference $r=1$ and with the general term $a_{n}=n$.
Arithmetic mean property. The terms $a_{n-1}, a_{n}, a_{n+1}$ of the arithmetic sequence satisfy the arithmetic average

$$
a_{n}=\frac{a_{n-1}+a_{n+1}}{2}
$$

In other words $a_{n}$ term is arithmetic mean of the previous $a_{n-1}$ term and next one $a_{n+1}$. Indeed, we have

$$
\frac{a_{n-1}+a_{n+1}}{2}=\frac{\left(a_{0}+(n-1) r\right)+\left(a_{0}+(n+1) r\right)}{2}=\frac{2 a_{0}+2 n r}{2}=a_{n}
$$

Distance property The following equalities hold

$$
a_{0}+a_{n}=a_{k}+a_{n-k}
$$

for $k=0,1,2, \ldots n$;
Namely, we see that for every $k=0,1,2, \ldots, n$,

$$
a_{k}+a_{n k}=\underbrace{a_{0}+k r}_{a_{k}}+\underbrace{a_{0}+(n k) r}_{a_{n k}}=a_{0}+\underbrace{a_{0}+n r}_{a_{n}}=a_{0}+a_{n} .
$$

Example 5.1 Show that the following sequences are arithemetic.

$$
\text { (i) } \quad a_{n}=\frac{3 n+1}{3}, \quad n=0,1,2, \ldots ;
$$

$$
\text { (ii) } \quad a_{n}=1+n^{2}, \quad n=0,1,2, \ldots, \text { : }
$$

Solution (i). We check if the difference $r=a_{n+1}-a_{n}$ of successive terms of this sequence is constant, i.e. it is independent of $n$.

$$
r=a_{n+1}-a_{n}=\underbrace{\frac{3(n+1)+1}{3}}_{a_{n+1}}-\underbrace{\frac{3 n+1}{3}}_{a_{n}}=\frac{3 n+1}{3}+\frac{1}{3}-\frac{3 n+1}{3}=\frac{1}{3}, \quad n=0,1,2, \ldots ;
$$

Answerx: The sequence ( $i$ ) is arithmetic. The difference between two consecutive terms of the sequence is the constant $r=\frac{1}{3}$ and does not depend on $n=0,1,2, \ldots$;
Solution (ii). We check if the difference $r=a_{n+1}-a_{n}$ of successive terms of this sequence is constant, i.e. it is independent of $n$.

$$
r=a_{n+1}-a_{n}=\underbrace{1+(n+1)^{2}}_{a_{n+1}}-\underbrace{\left(1+n^{2}\right)}_{a_{n}}=1+n^{2}+2 n+1-\left(1+n^{2}\right)=1+2 n,
$$

Answerx: We can see that the sequence (ii) is not arithmetic one, because the difference between two successive terms of the sequence

$$
r=2 n+1
$$

for $n=0,1,2, \ldots ;$ depends on $n$.

### 5.1.1 Questions

Question 5.1 Show that the following sequence is artmetic

$$
\begin{aligned}
& \text { (i) } \quad a_{n}=\frac{8 n+1}{5}, \quad n=0,1,2, \ldots \\
& \text { (ii) } a_{n}=1+2^{n}, \quad n=0,1,2, \ldots,
\end{aligned}
$$

### 5.2 Arithmetic series.

Arithmetic progress is the sum of terms of an arithmetic sequence. We write as below

$$
a_{0}+a_{1}+a_{2}+\cdots+a_{n}
$$

or

$$
a_{0}+\left(a_{0}+r\right)+\left(a_{0}+2 r\right)+\cdots+\left(a_{0}+n r\right)
$$

In sigma notation we write the arithmetic series as the following sum:

$$
\sum_{k=0}^{n} a_{k}=a_{0}+a_{1}+a_{2}+\cdots+a_{n}
$$

or

$$
\sum_{k=0}^{n}\left(a_{0}+k r\right)=a_{0}+\left(a_{0}+r\right)+\left(a_{0}+2 r\right)+\cdots+\left(a_{0}+n r\right)
$$

In order to derive a formula for the sum of $n$ terms of an arithmetic sequence. let's consider the sum

$$
S_{n}=a_{0}+a_{1}+a_{2}+\cdots+a_{n-1}+a_{n} .
$$

snd in reverse order of components

$$
S_{n}=a_{n}+a_{n-1}+\cdots+a_{2}+a_{1}+a_{0}
$$

By adding left and rigth sides we get formula

$$
2 S_{n}=\left(a_{0}+a_{n}\right)+\left(a_{1}+a_{n-1}\right)+\left(a_{2}+a_{n-2}\right)+\cdots+\left(a_{n-1}+a_{1}\right)+\left(a_{n}+a_{0}\right)
$$

Because terms of arithmetic progression satisfy the equalities

$$
a_{0}+a_{n}=a_{1}+a_{n-1}=a_{2}+a_{n-2}=\cdots=a_{n}+a_{0}
$$

therefore, we easily find formula for two sums

$$
2 S_{n}=(n+1)\left(a_{0}+a_{n}\right)
$$

or

$$
S_{n}=\frac{(n+1)\left(a_{0}+a_{n}\right)}{2}
$$

Example 5.2 Find the formula for the sum of arithmetic progress

$$
1+2+3+\cdots+n
$$

Solution. Let us see that in this arithmetic progression the first term $a_{0}=0$ and the difference $r=1$.
Using the above formula, we find the sum

$$
S_{n}=\frac{(n+1)}{2}\left(2 a_{0}+n r\right)=\frac{(n+1) n}{2} .
$$

Question 5.2 Find the sum of $n$ terms of the sequence with a general term

$$
a_{n}=\frac{3 n+5}{2}, \quad n=0,1,2, \ldots,
$$

### 5.3 Geometric sequences and geometric series

Let us consider a geometric sequence of numbers with terms

$$
a_{0}, \quad a_{1}=a_{0} * q, \quad a_{2}=a_{0} * q^{2}, \quad \ldots, \quad a_{n}=a_{0} * q^{n}, \quad n=0,1,2, \ldots ;
$$

where $a_{0}$ is the first term and the quotient $q=\frac{a_{k+1}}{a_{k}}, \quad k=0,1,2, \ldots, n-1$

The general term of the sequence

$$
a_{n}=a_{0} q^{n}, \quad n=0,1,2, \ldots ;
$$

For example, the sequence of numbers

$$
1,2,2^{2}, 2^{3}, \ldots, 2^{n} \ldots ;
$$

has of the seris form

$$
S_{\infty}=+2+3+\ldots+2^{n}+\ldots
$$

with the first term $a_{0}=1$, the quotient of $q=2$ and the general term $a_{n}=2^{n}$. This sequence is a geometric one.
Note that when the quotient $q=0$ then the geometric sequence is constant $a_{n}=a_{0}$ for every $n=1,2, \ldots$;
Geometric mean. Note that the absolute value of terms of a geometric sequence is the geometric mean of the previous $a_{n-1}$ term and the next one $a_{n+1}$.

$$
\left|a_{n}\right|=\sqrt{\left|a_{n-1} a_{n+1}\right|}
$$

Indeed, we have

$$
a_{n-1} * a_{n+1}=a q^{n-1} * a * q^{n+1}=a^{2} * q^{2 n}=a_{n}^{2}
$$

Hence the geometric mean

$$
\left|a_{n}\right|=\sqrt{a_{n-1} * a_{n+1}}
$$

Also the products of two sequence terms equally distant by $k$ from $a_{0}$ and by $k$ from $a_{n}$ are equal for every $k=0,1,2, \ldots n$;
Namely, we check that for every $k=0,1,2, \ldots, n$

$$
a_{k} * a_{n-k}=\underbrace{a_{0} * q^{k}}_{a_{k}} * \underbrace{a_{0} * q^{n-k}}_{a_{n-k}}=\underbrace{a_{0}\left(a_{0} q^{n}\right)}=a_{0} * a_{n} .
$$

Example 5.3 Check if the following geometric sequences are geometric

$$
\text { (i) } \quad a_{n}=\frac{3^{n}}{2^{n}}, \quad n=0,1,2, \ldots
$$

(ii) $a_{n}=n^{2}, \quad n=1,2, \ldots$, :

Solution (i). We check if the quotient of $q$ consecutive terms of the sequence is constant,i.e. independent of $n$

$$
\frac{a_{n+1}}{a_{n}}=\left(\frac{3^{n+1}}{2^{n+1}}\right):\left(\frac{3^{n}}{2^{n}}\right)=\frac{3^{n+1} * 2^{n}}{2^{n+1} * 3^{n}}=\frac{3}{2}=q, \quad n=0,1,2, \ldots
$$

Answer: The sequence $(i)$ is geometric because the constant quotient $q=\frac{3}{2}$ for all $n=$ $0,1,2, \ldots$;
Solution (ii).
We check if the quotient of $q$ of successive terms of the sequence depends on $n$.

$$
q=\frac{a_{n+1}}{a_{n}}=\frac{(n+1)^{2}}{n^{2}}=\frac{n^{2}+2 n+1}{n^{2}}=1+\frac{2}{n}+\frac{1}{n^{2}}, \quad n=1,2, \ldots ;
$$

Answer The sequence is not geometric, because the quotient $q=1+\frac{2}{n}+\frac{1}{n^{2}}$ depends on $n$.

Example 5.4 Of the four numbers $x, y, z, t$, the first three form an arithmetic sequence, the sum of which is equal to 12 , and the last three form a geometric sequence whose sum equals 15. Find the numbers

Solution. We know that the sum of the first three numbers is 12 and the sum of the last three numbers is 19. So we have two linear equations

$$
\begin{aligned}
& x+y+z=12 \\
& y+z+t=19
\end{aligned}
$$

The first three numbers form an arithmetic sequence, so the difference between the second number y and the first x is the same as the difference between the third z and the second y

$$
y-x=z-y
$$

Similarly, the last three numbers form a geometric sequence, so the quotient of the third number $z$ and the second number $y$ is the same as the quotient of the fourth number $t$ and the third number $z$

$$
\frac{z}{y}=\frac{t}{z}
$$

This way you have a system of four equations with four unknowns

$$
\begin{array}{rlc}
x+y+z & = & 12 \\
y+z+t & = & 19 \\
y-x & = & z-y \\
\frac{z}{y} & = & \frac{t}{z} t=\frac{2 z^{2}}{x+z}
\end{array}
$$

We find the solution of this system using the Gaussian elemination method. Namely, from the last two equations we find

$$
y=\frac{x+z}{2} \quad \text { and } \quad t=\frac{2 z^{2}}{y+z}
$$

After substituting into the first two equations and reducing the terms, similar we will get equations

$$
\begin{aligned}
& x+z=8 \\
& z^{2}+4 z-60=0
\end{aligned}
$$

The quadratic equation has two real roots

$$
z_{1}=9, \quad z_{2}=-10
$$

From where we find two solutions

$$
x_{1}=2, \quad y_{1}=4, \quad z_{1}=6, \quad t_{1}=9
$$

and

$$
x_{2}=18, \quad y_{2}=4, \quad z_{2}=-10, \quad t_{2}=25
$$

Example 5.5 $Z$ czterech liczb $x, y, z, t$ trzy pirwsze tworz ciag arytmetyczny, których suma równa jest 12, a trzy ostatnie tworza ciagg geometryczny, których suma równa jest 15 . Znajdź teliczby

Rozwiązanie. Wiemy, że suma trzech pierwszych liczb równa jest 12, a suma trzech ostatnich liczb równa jest 19. Zatem mamy dwa równania liniowe

$$
\begin{aligned}
& x+y+z=12 \\
& y+z+t=19
\end{aligned}
$$

Trzy pierwsze liczby tworza̧ cia̧g arytmetyczny, dlatego różnica pomiȩdzy druga̧ liczba̧ y i pierwszą x jest ta sama jak różnica pomiȩdzy trzecią z i drugą y $y-x=z-y$ Podobnie trzy ostatnie liczby tworzą cią geometryczny, dlatego iloraz trzeciej z i liczby drugie y jest taki sam iloraz czwarej liczby t i trzeciej liczby z

$$
\frac{z}{y}=\frac{t}{z}
$$

W ten sposób otrzymaliśmy układ czterech równań z czterema niewiadomymi

$$
\begin{array}{rcc}
x+y+z & = & 12 \\
y+z+t & = & 19 \\
y-x & = & z-y \\
\frac{z}{y} & = & \frac{t}{z} t=\frac{2 z^{2}}{x+z}
\end{array}
$$

Rozwia̧zanie tego układu znajdujemy metoda̧ eleminacj Gaussa. Mianowicie, z ostatnich dwóch równań znajdujemy

$$
y=\frac{x+z}{2} \quad \text { oraz } \quad t=\frac{2 z^{2}}{y+z}
$$

Po podstawieniu do pierwszych dwóch równań i redukcji wyrazw podobny otrzymamy równania

$$
\begin{aligned}
& x+z=8 \\
& z^{2}+4 z-60=0
\end{aligned}
$$

Równanie kwadratowe ma dwa pierwiastki rzeczywiste $z_{1}=9, \quad z_{2}=-10$. Skąd znajdujemy dwa rozwiązania

$$
x_{1}=2, \quad y_{1}=4, \quad z_{1}=6, \quad t_{1}=9 \quad \text { and } \quad x_{2}=18, \quad y_{2}=4, \quad z_{2}=-10, \quad t_{2}=25
$$

### 5.3.1 Questions

Question 5.3 Check that the following sequences is geometric.

$$
\begin{array}{ll}
\text { (i) } & \frac{(\sqrt{2})^{n}}{5^{n}}, \ldots \\
& n=0,1,2, \ldots \\
\text { (ii) } & \sqrt{n}, \ldots
\end{array} \quad n=1,2, \ldots ;
$$

Question 5.4 Give the first term and the $n_{t h}$ term of the sequence Find the quotient of this geometric sequence
(i) $\frac{1}{5}, \frac{3}{5}, \frac{3^{2}}{5}, \frac{3^{3}}{5}, \ldots$;
(ii) $\sqrt{2}, 2,2 \sqrt{2}, 4,4 \sqrt{2}, \ldots$ :

Question 5.5 Of the four numbers $m, n, k, l$, the first three form an arithmetic sequence, the sum of which is equal to 3 , and the last three form a geometric sequence whose sum equals 9. Find these numbers

## Chapter 6

## Prime numbers. Euclidean Algorithm

The number 2 is the only smalest even and prime number


Number line. 1 is not a prime number

### 6.1 Introduction

One of the most important operations on numbers is the decomposition of a natural number into prime number factors. We find the decomposition of numbers into prime factors on the basis of the fundamental theorem of arithmetic.
The direct consequence of decomposing natural numbers into prime factors is finding the greatest common divisor (GCD) and the least common multiple of two natural numbers. One of the optimal algorithms for determining the greatest common divisor of two natural numbers is Euclid's algorithm.

### 6.2 Prime numbers

Let ua start with definition of prime numbers
Definicja 6.1 The natural number $p>1$ is called a prime number, if number $p$ is not divisible by a different from 1 and itselfp This means that prime numbers do not have divisors different than 1 and itself. Every other number than prime numbers is called composite number.

Let us note that the natral number $p=1$ is not a prime number. Since it has only one divisor equal 1, not greater than 1. The number 0 is also not prime because it is less than 1 and 0 has many divisors. Zero divided by any natural number other than zero is equal to 0 . Let's list a few more primes

$$
2,3,5,7,11,13,17,19,23,29,31.37 .41 .43 .47 .51 .53 .59 .61 \ldots ;
$$

It is obvious from the definition that the number $p=2$ is the only even prime. The set of primes is not closed with respect to four arithmetic operations.

Example 6.1 The numbers $m=7$ and $n=3$ are prime, but their sum $m+n=7+3=10$ is not prime and also the difference $7-3=4$ is not prime. Likewise, the product of these numbers $m * n=3 * 7=21$ is not a prime number.

One of the most important properties of prime numbers is the following theorem:
Theorem 6.1 Fundamental theorem of arithmetic . For every natural number $n$ exist prime numbers

$$
p_{1}, p_{2}, p_{3} \cdots, p_{k}
$$

such that

$$
n=p_{1} * p_{2} * p_{3} * \cdots * p_{k}
$$

Moreover primes $p_{1}, p_{2}, \ldots, p_{k}$ are uniqualy determinded for a given natural $n$.

### 6.3 Factorisation of natural numbers

From fundamental theorem of arithmetic, we conclude that every positive natural number can be factorised in primes. It means that every positive natural number $m>1$ breaks down into the product of prime numbers. Moreover, this distribution of the number $m$ is only one, there are no other prime factors.
Let us consider one of simple methods of factorization natural numbers. Namely, we divide natural number $m$ by successive prime numbers. Then the number $m$ is equal to the product of its divisors.

Example 6.2 Factorise $m=1638$ into prime factors.
We followthe scheme

| 1638 | 2 |
| :--- | :--- |
| 819 | 3 |
| 273 | 3 |
| 91 | 7 |
| 13 | 13 |
| 1 |  |

The number 1638 is factored by $2,3,3,7,13$
Thus

$$
1638=2 * 3 * 3 * 7 * 13
$$

Example 6.3 Factorise the number $m=5040$ into prime factors. In order to factorise number $m$ we follow the scheme

| 5040 | 2 |
| :--- | ---: |
| 2520 | 2 |
| 1260 | 2 |
| 630 | 2 |
| 315 | 3 |
| 105 | 5 |
| 21 | 3 |
| 7 | 7 |
| 1 |  |

Hence, we get the distribution of number $m=5040$ by primes $2,2,2,2,3,5,3,7$,

$$
5040=2 * 2 * 2 * 2 * 3 * 5 * 3 * 7
$$

Let us note that 5040 is equal to 7 !

$$
7!=1 * 2 * 3 * 4 * 5 * 6 * 7=5040
$$

Question 6.1 It is known that the natural number $m$ is divisible by 5 and is decomposes into three prime factors, which the sum is equal 14. Find the number $m$.

### 6.4 Greatest common divisor $G C D(a, b)$

The greatest common divisor of two natural numbers $a i b$ is denoted by the symbol $G C D(a, b)$. One of the ways of calculating the greatest common divisor of given natural numbers $a i b$ is the decomposition of these numbers into prime factors.
Let us consider some examples of calculating $G C D(a, b)$ by decomposing $a$ and $b$ into prime factors

Example 6.4 Let be given number $a=21$ and the number $b=57$. The distribution of these numbers is obvious

$$
21=3 * 7 \quad i \quad 57=3 * 19
$$

The common divisor of numbers 21 and 57 is 3 because 3 divides 21 and divides 57. Also, these numbers do not have other common divisors.
Thus we get the value of the greatest common divisor

$$
G C D(21,57)=3
$$

Example 6.5 Find the greatest common divisor of 42 and 78.
We factoris both numbers into prime factors according to the scheme

| 42 | 2, | 78 | 2 |
| ---: | ---: | ---: | ---: |
| 21 | 3, | 39 | 3 |
| 7 | 7, | 13 | 13 |
| 1 |  | 1 |  |

Where do we get the number distribution

$$
42=2 * 3 * 7 \text { and } 78=2 * 3 * 13
$$

The common factors for these numbers are 2 and 3. Therefore, the greatest common divisor of 42 and 78 is $G C D(42,78)=2 * 3=6$.

Consider one more example of determining the greatest common divisor by the distribution of numbers into prime factors

Example 6.6 Find the greatest common divisor of 210 and 231

| 210 | 2, | 231 | 3 |
| ---: | ---: | ---: | ---: |
| 105 | 3, | 77 | 7 |
| 35 | 7, | 11 | 11 |
| 5 | 5 | 1 |  |
| 1 |  |  |  |

Where do we get the distribution of 210 and 231 into prime factors

$$
210=2 * 3 * 7 * 5 \quad \text { and } \quad 231=3 * 7 * 11
$$

Common divisors for all numbers are 3and 7. Therefore, the greatest common divisor of 210 and 231 is $G C D(210,231)=3 * 7=21$.
We check that $G C D(210,231)=21$ divides 210 and 231

$$
210: 21=10 \text { oraz } 231: 21=11
$$

### 6.5 Euclidean Algorithm (325-265 B.C.)

The most effective way to determine the greatest common divisor of two natureal numbers is the Euclidean algorithm. Already in ancient times in Egypt, Greek Euclid teacher and dean of the Faculty of Natural Sciences at the University of Alexandria he appliied an algorithm for finding the greatest common divisor of two natural numbers. From that time the algorithm is called as Euclidean Algoritm .
Below we present a description of the Euclid's algorithm.
Euclid's algorithm. The greatest common divisor of given natural numbers $a$ and $b$ we calculate by creating terms of a decreasing sequence of divisin withremainders

$$
r_{0}>r_{1}>r_{2}>r_{4}>\ldots>r_{n-1}>r_{n}
$$

starting from given numbers $r_{0}=a$ and $r_{1}=b$, according to the algorithm

$$
\begin{array}{lll}
\frac{r_{0}}{r_{1}} & =k_{2}+\frac{r_{2}}{r_{1}}, & r_{2}=r_{0}-k_{2} * r_{1} \\
\frac{r_{1}}{r_{2}} & =k_{3}+\frac{r_{3}}{r_{2}}, & r_{3}=r_{1}-k * 3 * r_{2} \\
\frac{r_{2}}{r_{3}} & =k_{4}+\frac{r_{4}}{r_{3}}, & r_{4}=r_{2}-k_{4} * r_{3} \\
\cdot & \cdots \cdots \cdots \cdots & \\
\frac{r_{n-1}}{r_{n}}= & k_{n+1}+\frac{r_{n+1}}{r_{n}}, & r_{n}=r_{n-2}-k_{n} * r_{n-1}
\end{array}
$$

Note that the above formulas can be written in one recursive formula

$$
\begin{equation*}
r_{i}=r_{i-2}-k_{i} * r_{i-1}, \quad \text { for } \quad r_{0}=a, \quad r_{1}=b, \quad ; i=2,3, \ldots, n, \tag{6.1}
\end{equation*}
$$

where the coefficients $k_{i}=E\left[\frac{r_{i}}{r_{i+1}}\right] . i=2,3,4, \ldots$ are integers. ${ }^{1}$
The last non-zero term of the sequence $r_{n} \neq 0$ is the greatest common divisor of the first two terms $r_{0}=a, r_{1}=b$. This results from the recursive formula (6.2).

$$
r_{n}=G C D\left(r_{0}, r_{1}\right)=G C D(a, b)
$$

Namely, if the number $d=G C D\left(r_{0}, r_{1}\right)$ is a divisor of the numbers $r_{0}$ and $r_{1}$, then $d$ is also a divisor of each subsequent remainder

$$
\begin{aligned}
& r_{2}=r_{0}-k_{2} * r_{1} \\
& r_{i}=r_{i-2}-k_{i} * r_{i-1}, \quad \text { for } i=2,3, \ldots, n,
\end{aligned}
$$

[^1]Euclid's algorithm. Let us apply Euclid's algorithm to calculate successive terms of the descending sequence of natural numbers

$$
r_{9}>r_{1}>r_{2}>r_{4}>\ldots>r_{n-1}>r_{n}
$$

starting with $r_{0}$ and $r_{1}$ and using the recursive formula

$$
\begin{align*}
& r_{2}=k_{2} * r_{0}-r_{1}  \tag{6.2}\\
& r_{i}=k_{i} * r_{i-2}-r_{i-1}, \text { for } i=2,3, \ldots, n,
\end{align*}
$$

Here, the factor $k_{i}=E\left[\frac{r_{i}}{r_{i+1}}\right] .{ }^{2}$
The last term in $r_{n} \neq 0$ not equal to zero is the greatest common divisor of the first two terms $r_{0}, r_{1}$,

$$
r_{n}=G C D\left(r_{0}, r_{1}\right) .
$$

This follows from the recursive formula (6.2). Namely, if the number $d=G C D\left(r_{0}, r_{1}\right)$ is a divisor of $r_{0}$ and $r_{1}$, then $d$ is also the divisor of each subsequent remainder calculated by formula

$$
\begin{aligned}
& r_{2}=r_{0}-k_{2} * r_{1} \\
& r_{i}=k_{i} * r_{i-2}+r_{i-1}, \text { for } i=2,3, \ldots, n,
\end{aligned}
$$

Calculation of the recursive sequence (6.2) we explaine on the examples. below

Example 6.7 Find the greatest common divisor of numbers $r_{0}=78 \quad i \quad r_{1}=42$ using the Euclid algorithm (6.2)

1. We divide the greater number $r_{0}=78$ by the lesser number $r_{1}=48$, according to the scheme

$$
\begin{array}{ll}
\frac{r_{0}}{r_{1}}=k_{2}+\frac{r_{2}}{r_{1}}, & r_{0}=k_{2} * r_{1}+r_{2} \\
\frac{78}{42}=1+\frac{36}{42}, & k_{2}=1,
\end{array}
$$

and calculate the remainder of $r_{2}$ from dividing $r_{0}=78$ by $r_{1}=42$

$$
\begin{aligned}
& r_{2}=r_{0}-k_{2} * r_{1}, \\
& r_{2}=78-1 * 42=36 .
\end{aligned}
$$

where $k_{2}=E\left[\frac{r_{0}}{r_{1}}\right]$ is the integer part of the fractgion $\frac{r_{0}}{r_{1}}$
3
2.Substitute $r_{1}=42, r_{2}=36$ and divide the number greater $r_{1}=42$ by the number less $r_{2}=36$, according to the scheme

$$
\begin{array}{ll}
\frac{r_{1}}{r_{2}}=k_{3}+\frac{r_{3}}{r_{2}}, & r_{1}=k_{3} * r_{2}+r_{3} \\
\frac{42}{36}=1+\frac{6}{36}, & k_{3}=1, \\
42=1 * 36+6 .
\end{array}
$$

[^2]and calculate the remainder of the division of the numbers $r_{1}=42$ and $r_{2}=36$
\[

$$
\begin{aligned}
& r_{3}=r_{1}-k_{3} * r_{2} \\
& r=42-36=6
\end{aligned}
$$
\]

3. Substitute $r_{2}=36, r_{3}=6$ and divide the greater number $r_{2}=36$ by the number less $r_{3}=6$, according to the scheme

$$
\begin{array}{ll}
\frac{r_{2}}{r_{3}}=k_{4}+\frac{r_{4}}{r_{3}}, & r_{2}=k_{4} * r_{3}+r_{4} \\
\frac{36}{6}=6, & k_{4}=6,
\end{array} \quad 36=6 * 6+0
$$

and calculate the remainder of the division of the numbers $r_{2}=42$ and $r_{3}=36$

$$
\begin{aligned}
& r_{4}=36-k_{4} * 6 \\
& r_{4}=36-36=0
\end{aligned}
$$

The greatest common divisor of 78 and 42 is the last remainder $r_{3}=6$ not zero, we write $G C D(78,36)=6$

We will consider the next example of using the Euclid algorithm (6.2).

Example 6.8 Find the greatest common divisor of $r_{0}=595$ and $r_{1}=204$.
As in the previous example, we find the greatest common divisor of numbers 595 and 204 using the formula (6.2)

$$
\begin{array}{cc}
r_{0}=595, r_{1}==204 & \text { resztar } \\
--\overline{595}=--\overline{187} & --- \\
\frac{18}{204}=2+\frac{204}{204} & r_{2}=187 \\
\frac{204}{187}=1+\frac{17}{204} & r_{3}=17 \\
\frac{187}{17}=11 & r_{4}=0
\end{array}
$$

The greatest common divisor of 595 and 204 is the last remainder $r_{3}=17$ non-zero, we write

$$
G C D(595,204)=17
$$

Example 6.9 Find the greatest common divisor of $r_{0}=1995$ and $r_{1}=1190$
As in the previous example, we find the greatest common divisor of numbers 1995 and 1190 using the formula (6.2)

$$
\begin{array}{ccc}
r_{0}=1995, r_{1}==1190 & \mid & \text { reszta } r \\
------ \\
\frac{1995}{1190}=1+\frac{805}{1190} & \mid & r_{2}=805 \\
\frac{1190}{805}=1+\frac{6}{385} & \mid & r_{3}=385 \\
\frac{805}{385}=2+\frac{35}{385} & \mid & r_{4}=35 \\
\frac{385}{35}=11 & \mid & r_{5}=0
\end{array}
$$

The greatest common divisor of 1995 and 1190 is the last remainder $r_{4}=35$ non-zero, we write $G C D(1995,1190)=35$

Example 6.10 Find the greatest common divisor of 975 and 690 numbers

## Solution.

We use the above-described Euclid's algorithm and calculate successive residuals

$$
\begin{array}{ccc}
a=r_{0}=975, \quad b=r_{1}=690 & \mid & \text { reszta } \\
------------------ \\
\frac{975}{690}=1+\frac{285}{690} & \mid & r_{2}=975-1 * 690=285 \\
\frac{690}{285}=2+\frac{120}{285} & \mid & r_{3}=690-2 * 285=120 \\
\frac{285}{120}=2+\frac{45}{120} & \mid & r_{4}=285-2 * 120=45 \\
\frac{120}{45}=2+\frac{30}{45} & \mid & r_{5}=120-2 * 45=30 \\
\frac{45}{30}=1+\frac{15}{30} & \mid & r_{6}=45-1 * 30=15 \\
\frac{30}{15}=2 & \mid & r_{7}=0
\end{array}
$$

The sequence of remainders

$$
975>690>285>120>45>30>15
$$

is descending.
The last remainder of the division $r_{6}=15$ nonzero is the greatest common divisor of numbers natural $r_{0}=975$ and ${ }_{1}=690$, we write

$$
G C D(975,690)=15
$$

Note that the greatest common divisor of $r_{6}=15$ of the numbers $r_{0}=975$ and $r_{1}=690$ is also the greatest common divisor of all previous residuals

$$
r_{2}=285, r_{3}=120, r_{4}=45, r_{5}=30, r_{6}=15
$$

4

[^3]
### 6.6 Least common multiple $\operatorname{LCM}(a, b)$

The common multiple of the given integers $a, b$ is the integer $m$, which is divisible by both $a$ and $b$. There are infinitely many common multiples of given natural numbers. We choose the smallest of them.

Example 6.1 For 5 and 7, the common multiple is $5 * 7=3535$ is the smallest common multiple of 5 and 7. The other common multiple of 5 and 7 is a number 70 because $70: 5=$ 24 and $70: 7=10$. However, 70 is not the lowest common multiple of 5 and 7.

The least common multiple is found by decomposing the given numbers into prime factors.
Example 6.2 Find the least common multiple of 120 and 210.
We break down 120 and 210 into prime factors according to the scheme

| 120 | 2, | 210 | 2 |
| ---: | :--- | ---: | :--- |
| 60 | 2, | $105 \mid$ | 3 |
| 30 | 2, | $35 \mid$ | 5 |
| $15 \mid$ | 3, | $7 \mid$ | 7 |
| $5 \mid$ | 5 | $1 \mid$ |  |
| $1 \mid$ |  | $\mid$ |  |

We choose common factors in the distribution of both numbers: 2,3 , and 5 . Then we add the factors to the product of $2 * 3 * 5$, which are not common, that is, they are not repeated. These are the 4 and 7 factors. The least common the product of these factors is a multiple

$$
2 * 3 * 5 * 4 * 7=1540
$$

Example 6.3 Find with the least common multiple of 910 and 1155
We decompose the numbers 910 and 1190 into prime factors according to the scheme

| 910 | 2, | $1155 \mid$ | 3 |
| ---: | :--- | ---: | ---: |
| $455 \mid$ | 5, | 385 | 5 |
| $91 \mid$ | 7, | $77 \mid$ | 7 |
| $13 \mid$ | 13, | $11 \mid$ | 11 |
| $1 \mid$ |  | $1 \mid$ |  |

We choose common factors in the distribution of both numbers: 5 and 7 . Then we add the factors to the product of $5 * 7$, which are not common, that is, they are not repeated. These are $2,3,11,13$ factors.
The least common multiple is the product of these factors

$$
5 * 7 * 2 * 3 * 11 * 13=30030
$$

Example 6.4 Find with the least common multiple and greatest common divisor of numbers 1065910 and 17765
We decompose the numbers 10659 and 17765 into prime factors according to the scheme

| 10659 | 3, | 17765 | 5 |
| ---: | :--- | ---: | ---: |
| $3553 \mid$ | 11, | $3553 \mid$ | 11 |
| $323 \mid$ | 17, | $323 \mid$ | 17 |
| $19 \mid$ | 19, | $19 \mid$ | 19 |
| $1 \mid$ |  | $1 \mid$ |  |

Hence, we find the least common multiple

$$
3^{5} * 11_{1} 7_{1} 9=53295
$$

and the geatest common divisor

$$
11 * 17 *!9=3553
$$

of numbers 10659 and 17765 .

### 6.6.1 Questions

Question 6.2 Factorize the given numbers into prime factors
(i) $a=184$
(ii) $\quad b=6006$

Question 6.3 Calculate the remainder after dividing the numbers given below

$$
\begin{equation*}
a=254 \quad i \quad b=15 \tag{i}
\end{equation*}
$$

(ii) $\quad b=2672 \quad i \quad b=848$

Question 6.4 List all pairs of prime numbers

$$
(p, \quad 2 p+1)
$$

from 1 to 50 .
Question 6.5 Check that for $p=3$ the numbers $p+10$ and $p+20$ are prime numbers. Prove that for a prime $p \neq 3$ at least one of the numbers $p+10$ or $p+20$ is a composite number.

Question 6.6 Find the greatest common divisor of 425 and 125
(i) by decomposing these numbers into prime factors.
(ii) using the Euclidean algorithm

Question 6.7 Find the greatest common divisor of numbers using Euclid's algorithm

$$
2672 \text { and } 848
$$

Question 6.8 Find integer solution of the system of equations

$$
\begin{aligned}
& x+y=180 \\
& G C D(x, y)=30
\end{aligned}
$$

5
Question 6.9 Find the least common multiple of numbers

$$
\begin{array}{llll}
\text { (i) } & 9 & \text { and } & 12 \\
\text { (ii) } & 36 & \text { and } & 48
\end{array}
$$

[^4]Question 6.10 Find the least common multiple of numbers
(i) 25 and 235
(ii) 512 and 5040
(iii) 333 and 555

Question 6.11 Check if, each prime number $p$ can be represented as a product of the difference and the sum of the natural numbers $a$ and $b$ ?

$$
p=(a-b)(a+b)
$$

Question 6.12 Prove that for every prime $p>4$ the number

$$
(p-1)(p+1)
$$

is divisible by 24

## Chapter 7

## Number representation in computer arithmetic

Rounding absolute error of a real number

$$
\epsilon_{x}=x-f l(x)
$$

Relative error of rounding $x \neq 0$

$$
\delta_{x}=\frac{x-f l(x)}{x}
$$

Rounding percentage error $x \neq 0$

$$
\delta_{x}^{\%}=\frac{x-f l(x)}{x} * 100 \%
$$

By rounding off the number x at the r -th decimal place, we add to the r -th digit 1 , if the next digit of x is greater than or equal to 5 , otherwise, we delete digits after the r-th. The operations of rounding the number $x$ on the r-th position are denoted by the symbol $f l_{r}(x)$.

Example 7.1 Let us round $\frac{22}{7}$ at the 5th decimal place.
We convert $\frac{22}{7}$ into a decimal fraction dividing the numerator 22 by the denominator 7 .

$$
\frac{22}{7}=22: 7=3.142857142857 \ldots
$$

As result operation of division, we get the number $3.142857142857 \ldots$; with infinite number of digits after the decimal point. On fifth decimal place of this number is digit $r=5$, followed by the next digit $7>5$ see here $3.142857142857 \ldots$; Then add 1 to the digit 5 to obtain rounded off number

$$
\frac{22}{7}=f l_{5}(3.142857142857 \ldots)=3.14286, \quad r=5
$$

### 7.1 Notation of numbers in floating decimal point

In calculations with computing systems and computers, numbers are written as floating decimal points

$$
x=\mp m 10^{c}, \quad m-\text { mantissa }, \quad c-\text { feature },
$$

where mantissa

$$
m=0 . \alpha_{1} \alpha_{2} \ldots \alpha_{r} ; \alpha_{1} \neq 0 ; 0 \leq \alpha_{i} \leq 9 ; i=1,2, \ldots, r
$$

The most significant digit $a l p h a_{1} \neq 0$ is always Not zero.
Therefore, the mantissa $m$ satisfies the following inequality

$$
0.1 \leq m<1
$$

Clearly, the number $x$ can have exact floating point representations on a computer if its mantissa has a finite number of digits.

For example, $\frac{1}{4}$ has exact representation because its mantissa $m=0.25$ and the feature $c=0$.
By contrast, the mantissa of the number

$$
\frac{1}{3}=0.333 \ldots
$$

it has infinitely many digits $m=0.333 \ldots$, and does not have an exact computer representation.
Any number, even with an infinite number of digits, can be saved on the computer with accuracy of the rounding error of the mantissa at the r-th decimal place.

$$
\epsilon \leq 0 . \underbrace{000 \ldots 0}_{r-z e r} 5=0.510^{-r} .
$$

For example

$$
x=\frac{2}{3}=0.66666666666 \ldots
$$

Rounded to 4th decimal place $(r=4)$

$$
f l(x)=0.6667
$$

has rounding error $\epsilon=0.0000333 \ldots$

Question 7.1 Round the following numbers to the 3rd decimal place and write them with a floating decimal point

$$
2 \frac{3}{4}, \frac{29}{7},-\frac{238}{13} .
$$

### 7.2 The absolute error of rounding.

The absolute error of rounding the number $x$ written in a floating decimal point

$$
x=\mp m 10^{c}
$$

we call differences

$$
\epsilon_{x}=f l_{r}(x)-x
$$

This error satisfies the inequality

$$
\left|f l_{r}(x)-x\right| \leq \epsilon * 10^{c},
$$

where $\epsilon=0.510^{-r}$.
Let

$$
x=0.57367864 * 10^{2}, \quad r=3 .
$$

Then the absolute error of $x$ in the third decimal place of the round is

$$
\begin{gathered}
\left|f l_{3}\left(0.57367864 * 10^{2}\right)-0.57367864 * 10^{2}\right|= \\
\left|0.574 * 10^{2}-0.57367864 * 10^{2}\right|=0.032136<\frac{1}{2} 10^{-3} * 10^{2}=0.05
\end{gathered}
$$

### 7.3 Rounding relative error.

Rounding relative error for a given number $x=\mp m 10^{c} \neq 0$ we define as follows:

$$
\delta_{x}=\frac{\epsilon_{x}}{x}=\frac{f l_{r}(x)-x}{x}, \quad \text { when } x \neq 0 .
$$

Since the mantissa $m \geq 0.1$, therefore the relative error satisfies the inequality

$$
\left|\frac{\mid f l_{r}(x)-x}{x}\right| \leq 0.5 * 10^{1-r}, \quad x \neq 0
$$

Indeed

$$
\begin{aligned}
\left|\frac{f l_{r}(x)-x}{x}\right| & =\left|\frac{f l_{r}\left(\mp m 10^{c}\right) \pm m 10^{c}}{\mp m 10^{c}}\right| \\
& \leq\left|\frac{0.5 * 10^{-r}}{\mp m}\right| \leq 10 \epsilon=0.5 * 10^{1-r}
\end{aligned}
$$

So the relative error does not exceed computer precision $\delta=\frac{1}{2} 10^{1-r}$.
For example, if $r=3$ then computer precision $\delta=\frac{1}{2} 10^{-2}$
Calculate the relative rounding error of a number $x=0.57367864 * 10^{2}$

$$
\left|\frac{f l(x)-x}{x}\right|=\frac{0.032136}{0.57367864 * 10^{2}}=0.0005601742 .
$$

The relative error is directly related to the percentage error. Namely, the percentage error is expressed as the formula

$$
p \%=100 * \delta_{x} \%=100 \frac{f l(x)-x}{x} \%, \text { when } x \neq 0
$$

We calculate the percentage error of $x=0.57367864 * 10^{2}$

$$
p \%=100 * 0.5601742 * 10^{-3} \%=0.5601742 * 10^{-1} \%=0.05601742 \% .
$$

The results of the computation in the computer of the four arithmetic operations $x \pm y, x y$ and division $x / y$ for generic are inaccurate even if $x$ and $y$ are exact data.
For example, let $x=0.11111111$ and $y=0.55555555$ be 8 -digit numbers in 8 -digit computer arithmetic (8-digit mantissa).
Note that the result of multiplication $x y=0.617283938271605 * 10^{-1}$ has a 15 th digit mantissa $m=0.617283938271605$ which is automatically rounded in the computer to the 8 digit mantissa 0.61728394 z absolute error epsilon $_{x}=0.000000018271605$.

Example 7.2 Calculate the value of an arithmetic expression in 3- and 5-digit arithmetic of numbers in floating-point notation

$$
\frac{2 \frac{1}{3} * 3 \frac{1}{7}+45.27}{4 \frac{2}{9}}
$$

Enter: absolute error, relative error, and percentage error of the calculation.
Solution. First, let's write the numbers $x=2 \frac{1}{3}, y=3 \frac{1}{7}, z=45.27, t=4 \frac{2}{9}$ in the form of a floating comma, then we round to the place $r=3$ and give the rounding error for each given number

$$
\begin{array}{lll}
x=2 \frac{1}{3}=2.33333 \ldots ; & f l_{3}(x)=0.233 * 10, & \epsilon_{x}=0.003333 \ldots \\
y=3 \frac{1}{7}=3.142857142857 \ldots ; & f l_{3}(y)=0.314 * 10, & \epsilon_{y}=0.0042857 \ldots ; \\
z=45.27 & f l_{3}(z)=0.453 * 10^{2}, & \epsilon_{z}=0.07 \\
t=4 \frac{2}{9}=4.222222 \ldots ; & f l_{3}(t)=0.422 * 10, & \epsilon_{t}=0.00222 \ldots ;
\end{array}
$$

Next, using the order of arithmetic operations, multiplication, division, addition and subtraction, let's calculate the value of the expression in 3-digits arithmetic:

$$
\begin{array}{ll}
\text { product } & =f l_{3}\left(2 \frac{1}{3}\right) * f l_{3}\left(3 \frac{1}{7}\right)=f l_{3}(2.33 * 3.14)=f l_{3}(7.3162)=7.32 \\
\text { sum } & =f l_{3}\left(7.32+f l_{3}(45.274)\right)=f l_{3}(7.32+45.3)=f l_{3}(52.62)=52.6 \\
\text { numerator } & =52.6, \quad \text { denominator } k=4.22, \\
\frac{\text { numerator }}{\text { denominator }} & =f l_{3}\left(\frac{52.6}{4.22}\right)=f l_{3}(12.4645)=12.5,
\end{array}
$$

Answerx: The value of the artmetic expression in 3 digit arithmetic is 12.5
Now let's calculate the value of this expression in 5-digit arithmetic.
We have the following data:

$$
\begin{array}{ll}
x=2 \frac{1}{3}=2.33333 \ldots ; & f l_{5}(x)=0.23333 * 10, \quad \epsilon_{x}=0.00003333 \ldots \\
y=3 f l_{5} \frac{1}{7}=3.142857142857 \ldots ; & f l_{5}(y)=0.31429 * 10, \quad \epsilon_{y}=0.000042857 \ldots ; \\
z=45.27 & f l_{5}(z)=0.4527 * 10^{2}, \quad \epsilon_{z}=0.0 \\
t=4 \frac{2}{9}=4.222222 \ldots ; & f l_{5}(t)=0.42222 * 10, \quad \epsilon_{t}=0.0000222 \ldots
\end{array}
$$

Similarly, let's calculate the value of an arithmetic expression in 5-digit arithmetic

$$
\begin{array}{ll}
\text { product } & =f l_{5}\left(2 \frac{1}{3}\right) * f l_{5}\left(3 \frac{1}{7}\right)=f l_{5}(2.3333 * 3.1429)=f l_{3}(7.333333)=7.3333 \\
\text { sum } & =f l_{5}\left(7.3333+f l_{5}(45.27)\right)=f l_{5}(7.3333+45.27)=f l_{5}(52.6033)=52.603 \\
\text { numerator } & =52.603, \quad \text { denominator }=4.2222, \\
\frac{\text { numerator }}{\text { denominator }} & =f l_{5}\left(\frac{52.603}{4.2222}\right)=f l_{3}(12.4587)=12.459
\end{array}
$$

Answer: The value of the artmetic expression calculated above in the 5 -digit arithmetic is 12, 459
Errors: The exact value of the expression: $=12,457$
Rounding absolute error in 3 digit arithmetic $12.5-12.457=0.043$.
Rounding relative error in 3 digit arithmetic $=\frac{0.043}{12.457}=0.00345 ; 0.345 \%$
Rounding absolute error in 5 digit arithmetic $12.459-12.457=0.002$.
Relative error of rounding in 5 digit arithmetic $=\frac{0.002}{12.457}=0.00016 ; 0.016 \%$.

### 7.4 Questions

Question 7.2 Round $\frac{157}{50}$ to the 3rd decimal place.

Question 7.3 Calculate the absolute error of the number 1.5782 rounded to the third decimal place.

Question 7.4 Calculate the absolute error of the sum of the numbers

$$
3.1415+1.5782
$$

rounded to the third decimal place.
Question 7.5 Calculate the absolute error of the product of numbers

$$
3.1415 * 1.5782
$$

rounded to the third decimal place.
Question 7.6 Calculate the relative error of the product of numbers

$$
3.1415 * 1.5782
$$

rounded to the third decimal place.
Question 7.7 Calculate the relative error of the number quotient

$$
\frac{3,1415}{1,5782}
$$

rounded to the third decimal place.
Question 7.8 Calculate the relative error of an arithmetic expression

$$
1.5782+3.1415=0.72345
$$

rounded to three decimal places.
Question 7.9 Calculate the relative error of an arithmetic expression

$$
\frac{1}{2}+\frac{1}{3}-\frac{1}{6}
$$

rounded to three decimal places.

Question 7.10 Calculate the relative error of an arithmetic expression

$$
1.5782+\frac{1}{3}
$$

rounded to three decimal places.
Question 7.11 Calculate the value of the following arithmetic expression in 3 and 5 digit floating point arithmetic

$$
\frac{2}{3}+2 \frac{1}{3}-\frac{1}{6}
$$

Provide errors: absolute, relative and percentage of calculations.
Question 7.12 Calculate the value of the following arithmetic expression in the 3rd and 5th digit arithmetic of numbers in the decimal notation

$$
\frac{7 \frac{2}{3} * 9 \frac{3}{7}+125.97}{3 \frac{7}{9}}+256.75
$$

Find the errors: absolute, relative and percentage of the calculation.

## Chapter 8

## Divisibility features of integers. Congruence and modulo operation

In this chapter we consider divisibility features and operations for dividing integers with remainders and congruence modulo operations.

### 8.1 Divisibility features of natural numbers

Divisibility features of natural numbers result from the general notation of numbers in the positional system. We are reminding, that in the decimal system, each n-digit number

$$
\begin{aligned}
m & =\alpha_{n-1} \alpha_{n-2} \cdots \alpha_{1} \alpha_{0} \\
& =\alpha_{n-1} * 10^{n-1}+\alpha_{n-2} * 10^{n-2}+\cdots+\alpha_{1} * 10^{1}+\alpha_{0} * 10^{0}
\end{aligned}
$$

where $0,1,2,3,4,5,6,7,8,9$ aredigits of $m$.
Now, let us formulate and give a simple proof of the feature of dividing a natural number by 3

### 8.1.1 Features of divisbility of natural numbers by 3 or by 9

Natural number

$$
m=\alpha_{n-1} \alpha_{n-2} \cdots \alpha_{1} \alpha_{0}
$$

is divisible by 3 if and only if sum of digits

$$
\alpha_{n-1}+\alpha_{n-2}+\cdots+\alpha_{1}+\alpha_{0}
$$

is divisible by 3 . Moreover, if the sum of the digits is divivisible by 9 , then $m$ is also divisible by 9 .
Let us consider a few examples.
Example 8.1 Let $m=$ 24. The digits of this two-digit number, when $n=2$, are $\alpha_{1}=2$ and $\alpha_{0}=4$
Sum of digits

$$
\alpha_{1}+\alpha_{0}=2+4=6
$$

is divisible by 3. So 24 is divisible by 3. Indeed

$$
24: 3=8
$$

Example 8.2 Let $m=381$. The digits of this three-digit number when $n=3$ are $\alpha_{2}=$ $3, \alpha_{1}=8$ and $\alpha_{0}=1$
Sum of digits

$$
\alpha_{2}+\alpha_{1}+\alpha_{0}=3+8+1=12
$$

is divisible by 3 because $12: 3=4$. So 381 is divisible by 3. Indeed

$$
381: 3=127
$$

Example 8.3 Let $m=5673$. The digits of this four-digit number $n=4$, are $\alpha_{3}=5, \alpha_{2}=$ $6, \alpha_{1}=7$ and $\alpha_{0}=3$ Sum of digits

$$
\alpha_{3}+\alpha_{2}+\alpha_{1}+\alpha_{0}=5+6+7+3=21
$$

is divisible by 3. So 5673 is divisible by 3. Indeed

$$
5673: 3=1891
$$

Example 8.4 Let $m=48,537$. Zithers of this five-digit number, when $n=5$, is $\alpha_{4}=$ $4, \alpha_{3}=8, \alpha_{2}=5, \alpha_{3}=7$ and $\alpha_{0}=7$
Sum of digits

$$
\alpha_{3}+\alpha_{2}+\alpha_{1}+\alpha_{0}=4+8+5+3+7=27
$$

is divisible by 3and 9. So 5673 is divisible by 3and 9. Indeed

$$
48537: 3=16177, \quad i \quad 48537: 9=5393
$$

Proof for 2-digit numbers. We write two-digit numbers in the form

$$
\alpha_{1} \alpha_{0}=\alpha_{1} * 10+\alpha_{0}
$$

Simple conversion of an algebraic expression

$$
\begin{aligned}
\alpha_{1} * 10+\alpha_{0} & =\alpha_{1} *(9+1)+\alpha_{0} \\
& =9 * \alpha_{1}+\left(\alpha_{1}+\alpha_{0}\right)
\end{aligned}
$$

it has a $9 * \alpha_{1}$ component with a factor of 9 , so that component is divisible by 3 and 9 .
Hence, we conclude that:
If the sum of the digits alpha $a_{1}+$ alpha $_{0}$ is divisible by 3 or 9 , then $m$ is divisible by 3 or 9 . The reverse statement is also true:

If $m$ is divisible by 3 or 9 then the sum of its digits alpha $a_{1}$ alpha $a_{0}$ te . $z$ is divisible by 3 or 9 .

We express these two sentences with one sentence:
The number $m$ is divisible by 3 or 9 if and only if its sum of the digits alpha $a_{1}+$ alpha $a_{0}$ is divisible by 3 or 9 .

This two-way relationship is called a necessary and sufficient condition. In this example it
is a necessary and sufficient condition of dividing $m$ by 3 or by 9 .

Let us repeat the proof of the divisibility of the number $m$ by 3 or by 9 for three-digit numbers.

Proof for 3-digit numbers. Let us write three-digit numbers as follows

$$
\alpha_{2} \alpha_{1} \alpha_{0}=\alpha_{2} * 100+\alpha_{1} * 10+\alpha_{0}
$$

The algebraic expression

$$
\begin{aligned}
\alpha_{2} * 100+\alpha_{1} * 10+\alpha_{0} & =\alpha_{2} *(99+1)+\alpha_{1} *(9+1)+\alpha_{0} \\
& =99 * \alpha_{2}+9 * \alpha_{1}+\left(\alpha_{2}+\alpha_{1}+\alpha_{0}\right)
\end{aligned}
$$

contains term $99 * \alpha_{2}+9 * \alpha_{1}$, which is divisible by 3 and 9 . Thus, if the sum of the digits $\alpha_{2}+\alpha_{1}+\alpha_{0}$ is divisible by 3 or 9 , then $m$ is also divisible by 3 or 9 .

Hence, we conclude that:
If the sum of the digits $\alpha_{2}+\alpha_{1}+\alpha_{0}$ is divisible by 3 or 9 , then $m$ is divisible by 3 or 9 .
The reverse statement is also true:
If $m$ is divisible by 3 or 9 then the sum of its digits $\alpha_{2}+\alpha_{1}+\alpha_{0}$ is also divisible by 3 or 9 .
We express these two sentences with one.
The number $m$ is divisible by 3 or 9 if and only if its sum of the digits $\alpha_{2}+\alpha_{1}+\alpha_{0}$ is divisible by 3 or by 9 .

This two-way relationship is called the necessary and sufficient condition of dividing $m$ by 3 or by 9 .

In the general case for $n$-digit numbers, the scheme of proof of the property of divisibility of $m$ by 3 or by 9 is the same as for two-digit and three-digit numbers.

Question 8.1 It is known that the natural number $m$ is divisible by 3 and has exactly 4 divisors, the sum of which is 128 . Find that number.

### 8.1.2 The feature of divisibility of a natural number by 5

It is very easy to recognize the number $m$ that is divisible by 5 or not.
A natural number $m$ is divisible by 5 if and only if its ones digits are 0 or 5 .
Example 8.5 We check that numbers

$$
30.35 .40 .45,150,155,2360,2,365,9800.9855 .9890 .9995
$$

are divisible by 5
Proof of the feature of dividing the number $m$ by 5 .
Let us consider a three-digit number $m$ that has a one digit 0 or 5 and let us write this number in the form

$$
\begin{aligned}
m & =\alpha_{2} * 10^{2}+\alpha_{1} * 10 \\
& =5 *\left(2 * \alpha_{2} * 10+2 * \alpha_{1}\right)
\end{aligned}
$$

or

$$
\begin{aligned}
m & =\alpha_{2} * 10^{2}+\alpha_{1} * 10+5 \\
& =5 *\left(2 * \alpha_{2} * 10+2 * \alpha_{1}+1\right)
\end{aligned}
$$

In the general case of n-digit numbers that have unity digit 0 or 5 , we also get the form with factor 5

$$
\begin{aligned}
m & =\alpha_{n-1} * 10^{n-1}+\alpha_{n-2} * 10^{n-2}+\cdots+\alpha_{1} * 10^{1} \\
& =5 *\left(2 * \alpha_{n-1} * 10^{n-2}+2 * \alpha_{n-2} * 10^{n-3}+\cdots+2 * \alpha_{1}\right)
\end{aligned}
$$

or

$$
m=5 *\left(2 * \alpha_{n-1} * 10^{n-2}+2 * \alpha_{n-2} * 10^{n-3}+\cdots+2 * \alpha_{2}+\alpha_{1}\right)
$$

From the above we conclude that the number $m$, which has the unity digit 0 or 5 is divisible by 5 .

### 8.2 Dividing numbers by 3 with remainder

Each natural number $m$ is divided by 3 or divided by 3 with the remainder of 1 or the remainder of 2 .

We write

$$
\begin{array}{llllllll}
m=3 k & \text { when the number } & m & \text { is divisible } & \text { by } 3 \\
m=3 k+1 & \text { when } & \text { the number } & m & \text { is divisible } & \text { by } 3, & \text { remainder } 1 \\
m=3 k+2 & \text { when number } & m & \text { is divisible } & \text { by } 3, & \text { remainder } 2
\end{array}
$$

Example 8.6 Let us divide two numbers with remainder

- $33: 3=11$ remainder 0
- $34: 3=11$ remainder $; 1$
- $35: 3=11$ remainder 2
or we write division as fractions
- $\frac{33}{3}=11$ remainder 0
- $\frac{34}{3}=11+\frac{1}{3} \quad$ remainder 1
- $\frac{35}{3}=11+\frac{2}{3} \quad$ remainder 2

Example 8.7 The sum of three consecutive numbers divisible by 3 is 36 . What are these numbers?

## Solution.

Let's write three more numbers that are divisible by 3

$$
3 k-3,3 k, 3 k+3
$$

The sum of these numbers

$$
(3 d-3)+3 k+(3 d+3)=9 d=36
$$

Hence, we calculate

$$
9 k=36, \quad k=36: 9 \quad k=4 .
$$

Answer:

$$
\begin{aligned}
& 3 k-3=3 * 4-3=9 \\
& 3 k=3 * 4=12 \\
& 3 k+3=3 * 4+1=15
\end{aligned}
$$

The consecutive numbers divisible by 3 , which the sum i 36 are

$$
9, \quad 12 \quad 15
$$

Indeed

$$
9+12+15=36
$$

Question 8.2 The sum of three consecutive numbers divisible by 3 is 72 . What are the numbers?

Question 8.3 The sum of three consecutive numbers divisible by 3 with the remainder 1 is 75. What are these numbers?

Question 8.4 The sum of three consecutive numbers divisible by 3 with the remainder 2 equals 105. What are these numbers?

### 8.3 Dividing numbers by 5 with remainder

Each natural number $m$ is divided by 5 or divided by 5 remainder 1 or remainder 2 or remainder 3 or remainder 4 .

Then we write

$$
\begin{array}{llllllll}
m=5 k & \text { when number } & m & \text { is divisible } & \text { by } 5 & \\
m=5 k+1 & \text { when } & \text { number } & m & \text { is divisible } & \text { by } 5, & \text { remainder } 1 \\
m=5 k+2 & \text { when } & \text { number } & m & i s & \text { divisible } & \text { by } 5, & \text { remainder } 2 \\
m=5 k+3 & \text { when } & \text { number } & m & \text { is divisible } & \text { by } 5, & \text { remainder } 3 \\
m=5 k+4 & \text { when } & \text { thenumber } & m & \text { is divisible } & \text { by } 5, & \text { remainder } 4
\end{array}
$$

Example 8.8 Do the division by 5 of the remainder

- $35: 5=7$ remainder 0
- $36: 5=7$ the remainder of 1
- $37: 5=7$ the remainder of the 2
- $38: 5=7$ the remainder of the 3
- $39: 5=7$ remainder 4
or we write division in the form of fractions
- $\frac{35}{5}=7$ the remainder of 0
- $\frac{36}{5}=7+\frac{1}{5}$ the remainder of 1
- $\frac{37}{5}=7+\frac{2}{5}$ remainder 2
- $\frac{38}{5}=7+\frac{2}{5} \quad$ remainder 3
- $\frac{39}{5}=7+\frac{2}{5}$ remainder 4

Example 8.9 The sum of three consecutive numbers divisible by 5 is 45. What are these numbers?

Let's write three consecutive numbers that are divisible by 5

## Solution.

$$
5 k-5,5 k, \quad 5 k+5
$$

The sum of these numbers

$$
(5 d-5)+5 k+(5 d+5)=15 d=45
$$

Where do we calculate $k$

$$
15 k=45, \quad k=45: 15 \quad k=3
$$

Hence, we find three consecutive numbers that are divisible by 5 , the sum of which is 45

$$
\begin{aligned}
& 5 k-5=5 * 3-5=10 \\
& 5 k=5 * 3=15 \\
& 5 k+5=5 * 3+5=20
\end{aligned}
$$

Consecutive numbers divisible by 5 , have the sum 45 are the following numbers

$$
10, \quad 15 \quad 20
$$

Indeed

$$
10+15+20=45
$$

Question 8.5 The sum of three consecutive numbers divisible by 5 with the remainder 1 is 108. What are these numbers?

## Solution.

Let's write three consecutive numbers that are divisible by 5 with a remainder of 1

$$
5 k+1,5 k+6, \quad 5 k+11
$$

The sum of these numbers

$$
(5 k+1)+(5 k+6)+(5 k+11)=15 k+18=108
$$

Where do we calculate $k$

$$
15 k=90, \quad k=90: 15 \quad k=6 .
$$

Hence, we find three consecutive numbers that are divisible by 5, the sum of which is 108

$$
\begin{aligned}
& 5 k+1=5=5 * 6+1+31 \\
& 5 k+6=5 * 6+6=36 \\
& 5 k+11=5 * 6+11=41
\end{aligned}
$$

Consecutive numbers divisible by 5 , which have the sum 45 are the following numbers

$$
31, \quad 36 \quad 41
$$

Indeed

$$
31+36+41=108
$$

Question 8.6 The sum of two consecutive numbers divisible by 5 with the remainder 2 is 79. What are these numbers?

Question 8.7 The sum of three consecutive numbers divisible by 5 with the remainder 3 is 129. What are these numbers?

### 8.3.1 General rule of divisibility of natural numbers with remainders

Each natural number $m$ is divided by the natural number $n$ with the remainder $r$. As a result of the division, we get the integer part $k$ and the remainder $r$. Then we write

$$
m: n=k+r: n \quad \text { or } \quad \frac{m}{n}=k+\frac{r}{n} \quad \text { or } \quad m=k * n+r
$$

where the remainder of $r=0,1,2, \ldots, n-1$.

The operation of dividing numbers results the integer part of dividing $m$ by the number $n$ with remainder $r$

- The integer part from division, we write $E[m: n]$ or $[m: n]$.

The value of function integer part $E[m: n]$ is equal to the largest integer not greater than $m: n$.

It means

$$
E[m: n] \leq m: n \quad \text { or } \quad[m: n] \leq m: n
$$

For example, let $m=37, \quad n=5$.

The largest integer in this division but not greater than

$$
37: 5=7 \frac{2}{5}
$$

is equal to 7 , wewrite

$$
E[37: 5]=E\left[\frac{37}{5}\right]=7 \quad \text { or } \quad ;[37: 5]=\left[\frac{37}{5}\right]=7
$$

Example 8.10 Calculate the integer part and the remainder of dividing by 6 numbers $m=36,37,38,3940,41$

## Solution:

$36: 6=6, \quad$ number 36 is divisible by $6, \quad$ remainder $0, \quad$ imteger part $k=6, r=0$.
$37: 6=6, \quad$ number 37 divides by $6, \quad$ remainder $1, \quad$ imteger part $k=6, r=1$,
$38: 6=6, \quad$ number 38 divides by $6, \quad$ remainder $2, \quad$ imteger part $k=6, r=2$,
$39: 6=6, \quad$ number 37 divides by $6, \quad$ remainder $3, \quad$ imteger part $k=6, r=3$,
$40: 6=6, \quad$ number 40 divides by $6, \quad$ remainder $4, \quad$ imteger part $k=6, r=4$,
$41: 6=6, \quad$ number 31 divides by $6, \quad$ remainder $5, \quad$ integer part $k=6, r=5$,
General formula for dividing a natural number $m$ by 6

$$
m=6 k+r, \quad \text { remainder } \quad r=0,1,2,3,4,5 .
$$

Question 8.8 Applying the general formula for dividing a natural number $m$ by 6 , show that each prime $p>3$ is divided by 6 with the remainder of 1 or the remainder of 5 . Note the number $p$ in the form

$$
p=6 * k+1, \text { or } p=6 k-1
$$

where $k$ is an integer.
Write prime number $p=7901$ in the form $p=6 k-1$

### 8.4 Congruent numbers

The integers $a$ and $b$ are called congruent modulo number $n$, if their difference $a-b$ is divisible by $n$. It means that $a$ and $b$ divided by $n$ have the same remainder $r$.
For example

| 13 | is congruent to | 3 modulo 2 | since difference | $(13-3): 2=5$ |
| :--- | :--- | :--- | :--- | :--- | is divisible by 2

Charles Gauss (1777-1835) introduced the modulo operation markings.

$$
a \equiv b(\bmod n)
$$

Let us note that if $a \equiv b(\bmod n)$ then

$$
a-b=k * n
$$

for some integer $k$.

Example 8.11 By writing

$$
27 \equiv 13(\bmod 7)
$$

we understand that the difference of $27-13$ is divisible by 7. In this example

$$
(27-13): 7=2 .
$$

This means that $27-13=2 * 7$ for $k=2$.

Example 8.12 Which of congruences are true?

$$
\begin{array}{lll}
7 \equiv 3(\bmod 2), & \text { true since } & (7-3): 2=4: 2=2 \text { isdivisibleby } 2 \\
12 \equiv 5(\bmod 4), & \text { is not true since } & (12-5): 4=1+\text { remainder } 3
\end{array}
$$

Modulo division of a number $m$ by a number $n$ is equal to the remainder $r$. We write this using relation of equality $=$

$$
r=m(\bmod n) \quad \text { or } \quad r=(m: n-E[m: n]) * n
$$

For example let

$$
m=7,37, \quad n=2,5
$$

we compute the value of the remainder appling modulo division

$$
\begin{aligned}
& r=7(\bmod 2)=1, \quad r=(7: 2-E[7: 2]) * 2=\left(7 \frac{1}{2}-2\right) * 2=1 \\
& r=37(\bmod 5)=2, \quad r=(37: 5-[37: 5]) * 5=\left(7 \frac{2}{5}-7\right) * 5=2 .
\end{aligned}
$$

Also we calculate the remainder of modulo division on numbers $m=25,37$, and $n=15,12$

$$
\begin{array}{llll}
r=25(\bmod 15)=10 & \text { since } & 25: 15=1+\text { remainde } & 10 \\
r=37(\bmod 12)=1 & \text { since } & 37: 12=3+\text { remainder } & 1
\end{array}
$$

The exact result

$$
\begin{aligned}
& \frac{25}{15}=1+\frac{10}{15} \\
& \frac{37}{12}=3+\frac{1}{12}
\end{aligned}
$$

Then we write

$$
\begin{array}{lllllll}
r=a(\bmod n), & 25(\bmod 15)=10, & w h e n & a=25, & n=15, & \text { remainder } & r=10 \\
r=a(\bmod n), & 37(\bmod 12)=1, & \text { when } & a=37, & n=12, & \text { remainder } & r=1
\end{array}
$$

Example 8.13 Calculate $47(\bmod 5)$
We calculate

$$
47: 5=9+\text { remainder } 2
$$

Answer:

$$
47(\bmod 5)=2
$$

Example 8.14 Calculate $\quad 123(\bmod 7)$
We calculate

$$
123: 7=17+\text { remainder } \quad 4
$$

Answer:

$$
123(\bmod 7)=4
$$

### 8.4.1 Modulo relation

The modulo relation $\equiv$ on integers has similar properties to the equality relation $=$. Below we present properties of the modulo operation

1. Symmetry property $a \equiv b(\bmod n) \quad$ and $\quad b \equiv a(\bmod n)$

Example 8.15 Let us consider the congruences

$$
\begin{aligned}
& 15 \equiv 3(\bmod 4) \quad \text { and } \quad 3 \equiv 15(\bmod 4) \\
& a=15, \quad b=3
\end{aligned}
$$

The numbers $a=15$ and $b=3$ are congruent with the natural number $n=4$ in both cases, because

$$
(15-3): 4=3 \quad \text { and } \quad(3-15): 4=-3
$$

## 2. Transitive modulo relation

If the numbers $a, b$ and $b, c$ are congruent modulo integer $n$,

$$
a \equiv b((\bmod n) \text { and } b \equiv c(\bmod n)
$$

then the numbers $a$ and $c$ are congruent modulo $n$.

$$
a \equiv c(\bmod n)
$$

Example 8.16 Let us consider two modulo operations

$$
\begin{array}{lcc}
20 \equiv 12(\bmod 4) & \text { and } & 12 \equiv 8(\bmod 4), \\
a=20, \quad b=12, & c=8, & n=4
\end{array}
$$

The numbers $a=20$ and $c=8$ are also congruent modulo 4 when

$$
20 \equiv 8(\bmod 4)
$$

because

$$
(20-8): 4=3
$$

3. Addition and multiplication of congruences

If the congruences are true

$$
a \equiv b(\bmod n) \quad \text { and } \quad c \equiv d(\bmod n)
$$

then the sum of the right and left sides of these congruences satisfies the relation

$$
a+c \equiv b+d(\bmod n)
$$

and the product of the right and left sides of these congruences satisfies the relation

$$
a \cdot c \equiv b \cdot d(\bmod n)
$$

Example 8.17 Let us consider two congruences

$$
\begin{array}{lrll}
15 \equiv 3(\bmod 4) & \text { and } & 3 \equiv 15(\bmod 4) \\
a=15, & b=3 & c=3, & d=15, \quad n=4
\end{array}
$$

The numbers 15 and 3 congruent modulo natural number $n=4$, since

$$
(15-3): 4=3 \quad \text { and } \quad(3-15): 4=-3
$$

Power of congruences. In order to exeplain power of congruences let us consider the example

## Example 8.18

$$
\begin{aligned}
& 9 \equiv 3(\bmod 2) \\
& a=9, \quad b=3, \quad n=2 .
\end{aligned}
$$

Multiplying the congruences by sides, we get

$$
9^{2} \equiv 3^{2}(\bmod 2), \quad 9^{3} \equiv 3^{3}(\bmod 2), \ldots, 9^{k} \equiv 3^{k}(\bmod 2)
$$

or

$$
81 \equiv 9(\bmod 2), \quad 729 \equiv 27(\bmod 2), \ldots ;, 9^{k} \equiv 3^{k}(\bmod 2)
$$

We check

$$
\left.(81-9): 2=36, \quad(729-27): 2=702: 2=351, \ldots,\left(9^{k}-3^{k}\right): 2=\right): 2
$$

The difference of $9^{k}-3^{k}$ is also divisible by 2 . Because the unity digits of $9^{k}$ and $3^{k}$ are odd. Namely, the unit digits of the number $9^{k}$ to

$$
1,9,1,9,1,9, \ldots .
$$

and the unit digits of $3^{k}$ to

$$
9,7,1,9,7,1, \ldots . .
$$

The difference of odd numbers is an even number.
So the number $9^{k}-3^{k}$ is divisible by 2 for each natural number $k=1,2,3, \ldots$;
So, the from the numbers $9^{k}$ and $3^{k}$ are consistent modulo 2.
Example 8.19 The value of arythmetic expresion

$$
43^{125}-33^{125}
$$

is divisible by 10 .

By raising congruent equality to the power of 125

$$
43 \equiv 33(\bmod 10)
$$

we get congruences

$$
43^{125} \equiv 33^{125}(\bmod 10)
$$

The number 43 corresponds to the number 33 modulo 10, since

$$
(43-33): 10=1
$$

Therefore, the number $43^{125}$ is congruent to number $33^{125}$ modulo 10. It means that the difference

$$
43^{125}-33^{125}
$$

is divisible by 10 .
Application of congruence to check divisibility of numbers we present in the following example

Example 8.20 Applying the property of multiplication congruence prove that the number $7^{246}+1$ is divisible by 10 .

Solution. Note that $7^{2}+1=50$ is divisible by 10 . This means that 49 is equal to -1 modulo 10. So we have

$$
49 \equiv-1(\bmod 10)
$$

Raising the congruences to the power of 123 , we get

$$
49^{123} \equiv(-1)^{123}(\bmod 10), \quad 7^{246} \equiv(-1)^{123}(\bmod 10)
$$

Hence we get the congruence

$$
7^{246} \equiv-1(\bmod 10)
$$

which means that the number $7^{246}+1$ is divisible by 10 .

### 8.4.2 Solution of linear congruences

General equstion of linear congruence

$$
a * x \equiv b(\bmod n)
$$

where the integers $a, b$ and $n$ are given, and x is unknown
To solve the linear congruence, it means to find all integers which, substituted for $x$, satisfy the congruences, that is to find all the integers $x$ for which the number $a * x$ matches the number $b$ modulo $n$.
The first question that arises is, as with other equations, how many solutions does linear congruence have? It can be expected in advance that a linear congruence may have

- one solution, i.e. there is only one integer $x_{0}$ congruent with the number $b$ modulo $n$ such that

$$
a * x_{0} \equiv b(\bmod n)
$$

- more than one solution, that is, there is a finite or even infinite number of integers $x_{1}, x_{2}, \ldots, x_{k}, \ldots$; which are congruent with the number $b$ modulo $n$. Meaning

$$
a * x_{k} \equiv b(\bmod n), \quad k=1,2,3, \ldots
$$

- congruence has no solutions.

The existence of a solution of linear congruence results from the following necessary and sufficient condition:

## Necessary and sufficient condition

Linear congruence

$$
a * x \equiv b(\bmod n)
$$

has a solution if and only if the greatest common divisor of $\operatorname{GCD}(a, n)$ of the numbers a and $n$ is the divisor of $b$, that is $G C D(a, b) \mid n$.
After reading the above introduction about linear congruences, you need to solve some congruences to learn how to solve them.
Example 8.21 Expand the congruences

$$
2 * x \equiv 3(\bmod 2)
$$

We check the necessary and sufficient condition for the existence of a solution to this congruence.
Greatest common divisor

$$
G C D(a, b)=G C D(2,2)=2
$$

does not divide

$$
b=3, \quad 2 \dagger 3
$$

So there is no solution to this congruence.
Example 8.22 Solve the congruences

$$
3 * x \equiv 6(\bmod 9)
$$

We check the necessary and sufficient condition for the existence of a solution to this congruence.
Greatest common divisor

$$
G C D(a, b)=G C D(3,9)=3
$$

divides the factor

$$
b=6, \quad 3 \mid 6, \quad 6: 3=2 .
$$

So there is a solution to this congruence.
By definition of congruence, we have an equation

$$
3 * x-6=9 * k
$$

for all integer values $k=0, \pm 1, \pm 2, \pm 3, \ldots$;
Where do we calculate the solution from

$$
3 * x=9 * k+6, \quad x_{k}=3 * k+2, \quad \text { for } \quad k=0, \pm 1, \pm 2, \pm 3, \ldots
$$

Indeed, substituting the solution

$$
x_{k}=3 * k+2, \quad \text { for } \quad k=0, \pm 1, \pm 2, \pm 3, \ldots
$$

to the congruence

$$
3 * x \equiv 6(\bmod 9)
$$

we will get

$$
3 *(3 * k+2) \equiv 6(\bmod 9)
$$

Where does the identity come from

$$
(9 * k+6-6): 9=9 * k, \quad 9 * k=9 * k
$$

for each integer value of $k=0, \pm 1, \pm 2, \pm 3, \ldots$;

### 8.5 Questions

Question 8.9 Prove that a two-digit natural number $m=\alpha * 10_{1}+\alpha_{0}$ is divisible by 7 if and only if the algebraic expression

$$
3 \alpha_{1}+\alpha_{0}
$$

is divisible by 7.
Question 8.10 Prove that a three-digit natural number $m=\alpha_{2} 10^{2}+\alpha_{1} 10+\alpha_{0}$ is divisible by 7 if and only if the algebraic expression

$$
3^{2} \alpha_{2}+3 \alpha_{1}+\alpha_{0}
$$

is divisible by 7 .
Question 8.11 In general, prove that the natural number ( $n+1$ )-digit

$$
m=\alpha_{n} 10^{n}+\alpha_{n-1} 10^{n-1}+\ldots+\alpha_{2} 10^{2}+\alpha_{1} * 10+\alpha_{0}
$$

is divisible by 7 if and only if the algebraic expression

$$
3^{n} \alpha_{n}+3^{n-1} \alpha_{n-1}+\ldots+3^{2} \alpha_{2}+3 \alpha_{1}+\alpha_{0}
$$

is divisible by 7.

Question 8.12 Give a necessary and sufficient condition for the number to be four-digit

$$
m=\alpha_{3} 10^{3}+\alpha_{2} 10^{2}+\alpha_{1} 10+\alpha_{0}
$$

was divisible by 11.
Question 8.13 Do division with the remainder (i) $52: 3$ (ii) $331: 3$ (iii) $830: 3$
Question 8.14 The sum of three consecutive numbers divisible by 3 with a remainder of 1 equals 376. What are these numbers?
Question 8.15 The sum of three consecutive numbers divisible by 5 with a remainder of 2 equals 167. What are these numbers?
Question 8.16 Calculate the integer part and the remainder when dividing numbers

$$
m=136,237,3381
$$

by $n=7$.
Find the general formula for dividing the number $m$ by 7 with the remainder $r$.

Question 8.17 Calculate
(i) $8+10(\bmod 4)$
(ii) $2+5(\bmod 7)$
(iii) $12(\bmod 7)+13(\bmod 8)$

Question 8.18 Add, subtract and multiply the congruences. Check the results of these operations.

$$
18 \equiv 10(\bmod 4)
$$

and

$$
25 \equiv 17(\bmod 4)
$$

## Chapter 9

## General principle for creating positional number systems

The general form of positional systems in which we write numbrs is a polynomial

$$
\begin{equation*}
\alpha_{n-1} \rho^{n-1}+\alpha_{n-2} \rho^{n-2}+\cdots+\alpha_{2} \rho^{2}+\alpha_{1} \rho+\alpha_{0}, \tag{9.1}
\end{equation*}
$$

where the natural number $\rho \geq 2$ is called the base of the number system. The coefficients $\alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_{1}, \alpha_{0}$ are called digits of the number system.
The digits

$$
0,1,2,3, \ldots, \rho-1
$$

of a number in a system with the base $\rho$ are also one-digit numbers, The number of digits depends on the base $\rho$ and is equal to $\rho$.
A real number $x$, we write as a sequence of digits

$$
x=\left(\alpha_{n-1} \alpha_{n-2} \ldots \alpha_{1} \alpha_{0}\right)_{\rho}
$$

In the case of the commonly used decimal system, we leave out the parentheses with the index $\rho$ and we write the decimal number without the parentheses as below

$$
x=\alpha_{n-1} \alpha_{n-2} \ldots \alpha_{1} \alpha_{0}
$$

### 9.1 Numbers in positional systems. Examples

In this section we present examples of notation of numbers in positional systems
Example 9.1 In the decimal system $\rho=10$. the number

$$
x=2 * 10+4=24
$$

we write without brackets $x=24$
Example 9.2 In binary system, $\rho=2$. The same number

$$
x=1 * 2^{4}+1 * 2^{3}+0 * 2^{2}+0 * 2^{1}+0 * 2^{0}=1100
$$

we write with brackets $x=(10000)_{2}$
Example 9.3 In the octal system $\rho=8$. The same number $x=24$, we write with brackets in the octal notation

$$
x=3 * 8+0 * 8^{0}=(30)_{8}
$$

### 9.2 Decimal system.

In the decimal system, when the base $\rho=10$, we write a number $x$ as value of a polynomial at 10

$$
p_{n}(10)=a_{n-1} 10^{n-1}+a_{n-2} 10^{n-2}+\cdots+a_{110}+a_{0} .
$$

where the coefficients are digits $\alpha_{0}, \alpha_{1}, \alpha_{2}, \ldots, \alpha_{n-1}$.
$\alpha_{0}$ is the number of units of the number $x$, the coefficient at power $\rho^{0}=10^{0}$.
$\alpha_{1}$ is the number of tens of $x$, the coefficient with the power $\rho^{1}=10$.
$\alpha_{2}$ is the number of hundreds of $x$, the coefficient with the power $\rho^{2}=10^{2}$.
$\alpha_{3}$ is the number of thousands of the number $x$, the coefficient with $\rho^{3}=10^{3}$.
$\alpha_{n-1}$ is the coefficient with the power of $\rho^{n-1}=10^{n-1}$.
The most significant digit is always greater than or equal to $1, \alpha_{n-1} \geq 1$.
The digits of the decimal system

$$
0,1,2,3,4,5,6,7,8,9
$$

are also single-digit numbers
We write two-digit numbers in the general form

$$
a_{1} * 10+a_{0}=a_{1} a_{0},
$$

where the coefficient $a_{1}$, at 10 , the units digit is the coefficient $a_{0}$

Example 9.4 The number $\quad x=57$
$5 * 10+7=57$
has the coefficient $a_{1}=5$, at 10 , the unity digit $a_{0}=7$.
We write three-digit numbers in general form

$$
a_{2} * 100+a_{1} * 10+a_{0}=a_{2} a_{1} a_{0}
$$

Then the three-digit number has the general form

$$
a_{2} * 10^{2}+a_{1} * 10^{1}+a_{0} * 10^{0}=a_{2} a_{1} a_{0}
$$

Example 9.5 Let $x=348$. Then in the polynomial notation, we write

$$
x=3 * 10^{2}+4 * 10^{1}+8 * 10^{0}=348
$$

where the digit $a_{2}=3$ at 100 , the digit $a_{1}=4$, at 10 , the unity digit $a_{0}=8$.
In general, we write n-digit numbers in the polynomial form

$$
a_{n-1} 10^{n-1}+a_{n-2} 10^{n-2}+a_{n-3} 10^{n-3}+\cdots+a_{110}+a_{0}=a_{n-1} a_{n-2} \ldots a_{1} a_{0},
$$

where the powers of 10 are:

$$
\begin{aligned}
& 10^{1}=\underbrace{10}_{1} \\
& 10^{2}=\underbrace{10 * 10}_{2} \\
& 10^{3}=\underbrace{10 * 10 * 10}_{3} \\
& \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
& \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
& 10^{n-3}=\underbrace{10 * 10 * 10 * \ldots * 10}_{n-3} \\
& 10^{n-2}=\underbrace{10 * 10 * 10 * \ldots * 10}_{n-2} \\
& 10^{n-1}=\underbrace{10 * 10 * 10 * \ldots * 10}_{n-1}
\end{aligned}
$$

Example 9.6 Let $n=4$, and the four-digit number $x=7831$.
We write number $x=7831$ in the polynomial form

$$
x=7 * 1000+8 * 100+3 * 10+1=
$$

7831 or

$$
x=7 * 10^{3}+8 * 10^{2}+3 * 10^{1}+1=7831
$$

where the digit $a_{3}=7$ at 1000, the digit $a_{2}=8$ at 100 , the digit $a_{1}=3$ at 10 and the unity digit digit $a_{0}=1$.

We perform arithmetic operations in the decimal system in the following order: multiplication, division, addition and subtraction.
This order of arithmetic operations can be changed by parentheses.
In arithmetic expressions with parentheses, we first calculate the value of the expressions in the parentheses. In this example, we do the addition and subtraction in parentheses first, followed by the multiplication and division

$$
\begin{aligned}
\underbrace{(12+13)}_{25} * 4-\underbrace{(15-6)}_{9}: 3 & =25 * 4-9: 3 \\
& =100-3=97
\end{aligned}
$$

The table of addition of numbers to the decimal positional system

|  | Dodawanie |  |  |  | dziesiȩtne |  |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :---: |
| + | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |  |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |  |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |  |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |  |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |  |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |  |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |  |
| 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |  |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |  |
| 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |  |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |  |

We explain decimal addition by examples
Example 9.7 Do the decimal addition of numbers 25 and 13
We add the numbers $25+13$ using the decimal addition table.

$$
\begin{array}{r}
25 \\
+\quad 13 \\
-- \\
38
\end{array}
$$

Example 9.8 Do add numbers 89 and 56
We add numbers $25+13$ using the decimal addition table.

$$
\begin{array}{r}
89 \\
+\quad 56 \\
-- \\
145
\end{array}
$$

The subtraction table in the decimal positional system

|  | Odejmowanie |  |  |  | dziesiętne |  |  |  |  |  |  |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | :---: |
| - | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |  |
| 1 | 0 | -1 | -2 | -3 | -4 | -5 | -6 | -7 | -8 | -9 |  |
| 2 | 1 | 0 | -1 | -2 | -3 | -4 | -5 | -6 | -7 | -8 |  |
| 3 | 2 | 1 | 0 | -1 | -2 | -3 | -4 | -5 | -6 | -7 |  |
| 4 | 3 | 2 | 1 | 0 | -1 | -2 | -3 | -4 | -5 | -6 |  |
| 5 | 4 | 3 | 2 | 1 | 0 | -1 | -2 | -3 | -4 | -5 |  |
| 6 | 5 | 4 | 3 | 2 | 1 | 0 | -1 | -2 | -3 | -4 |  |
| 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | -1 | -2 | -3 |  |
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | -1 | -2 |  |
| 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | -1 |  |
| 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |  |

We explain subtraction by examples
Example 9.9 Do decimal subtraction of 29 and 18
We substruct numbers 29-18 using the subtraction table.

$$
\begin{array}{r}
29 \\
-\quad 18 \\
-- \\
11
\end{array}
$$

Example 9.10 Do decimal subtraction of numbers 629 and 354
We perfom subtraction of $629-354$ using the subtraction table

$$
\begin{gathered}
629 \\
-\quad 354 \\
-- \\
275
\end{gathered}
$$

The multiplication table in the decimal positional system

|  | Mnożenie |  |  |  |  | dziesietne |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :---: |
| ${ }^{*}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |  |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |  |
| 2 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |  |
| 3 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |  |
| 4 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 |  |
| 5 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |  |
| 6 | 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 |  |
| 7 | 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 |  |
| 8 | 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 |  |
| 9 | 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 | 90 |  |
| 10 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |  |

We explain multiplication using examples
Example 9.11 Do decimal multiplication 49 and 15
We do multiplication of $49 * 15$ using the multiplication and addition tables

$$
\begin{gathered}
49 \\
* 15 \\
--- \\
245 \\
49 \\
--- \\
735
\end{gathered}
$$

Example 9.12 Do the multiplication of numbers 345 and 123
We multiply numbers $345 * 123$ in writing using the multiplication and addition tables

$$
\begin{gathered}
345 \\
* 123 \\
--- \\
1035 \\
690 \\
345 \\
--- \\
42435
\end{gathered}
$$

We explain the written division with examples
Example 9.13 Let us divide 345 by 5

$$
\begin{gathered}
69 \\
--- \\
345: 5 \\
-30 \\
-- \\
45 \\
45 \\
--- \\
=
\end{gathered}
$$

Example 9.14 Divide 1659 by 21
We do a written division of 1659:21.

### 9.3 Even and odd decimal numbers

Even integer numbers have unity digit 0 or 2 or 4 or 6 or 8 .
Even integers are divisible by 2 . We define them by formula

$$
n=2 * k, \quad \text { for } k=0, \pm 1, \pm 2, \pm 3, \ldots
$$

For example

$$
\begin{array}{ll}
k=0, & n=2 * 0=0 \\
k=1, & n=2 * 1=2 \\
k=2, & n=2 * 2=4 \\
\ldots & \cdots \cdots \\
k=8, & n=2 * 8=16 \\
k=26, & n=2 * 26=52 .
\end{array}
$$

Sum, difference, and product of even numbers is an even number For example:

$$
\begin{gathered}
a=8, \quad b=6 \\
a+b=8+6=14, \quad a-b=8-6=2, \quad a * b=8 * 6=48
\end{gathered}
$$

Odd decimal numbers. Odd numbers have unity digit 1 or 3 or 5 or 7 or 9 . For example, numbers
$121,133,135,157,179$
they have the unity digits respectively

$$
1,3,5,7,9
$$

Odd integer numbers are of general form

$$
n=2 * k+1, \quad \text { or } \quad n=2 * k-1, \quad \text { for } k=0, \pm 1, \pm 2, \pm 3, \ldots
$$

For example

$$
\begin{array}{llll}
k=0, & n=2 * 0+1=1, & \text { or } & n=2 * 0-1=-1 \\
k=1, & n=2 * 1+1=3, & \text { or } & n=2 * 1-1=1 \\
k=2, & n=2 * 2+1=5, & \text { or } & n=2 * 2-1=3 \\
\ldots & \cdots . . & & \\
k=8, & n=2 * 8+1=17, & \text { or } & n=2 * 8-1=15 \\
k=26, & n=2 * 26+1=53 & \text { or } & n=2 * 26-1=51
\end{array}
$$

The product of odd numbers is an odd number For example:

$$
5 * 7=35, \quad 7 * 11=77, \quad 9 * 15=105
$$

The sum or difference of two odd numbers is an even number. On the other hand, the sum or difference of an odd number and an even number is an odd number.

Question 9.1 The sum of three consecutive odd numbers is 51. Find these numbers.

## Solution:

Consecutive odd numbers are

$$
2 n+1, \quad 2 n+3, \quad 2 n+5
$$

Their sum

$$
(2 n+1)+(2 n+3)+(2 n+5)=6 n+9=51
$$

We calculate n:

$$
6 n+9=51, \quad 6 n=42, \quad n=42: 6=7
$$

Let's calculate three consecutive even numbers

$$
2 n+1=2 * 7+1=15, \quad 2 n+3=2 * 7+3=17, \quad 2 n+5=2 * 7+5=19
$$

Indeed

$$
15+17+19=51
$$

Question 9.2 The sum of five consecutive even numbers is 200. Find these numbers.

## Solution:

The consecutive even numbers are

$$
2 n-4, \quad 2 n-2, \quad 2 n, \quad 2 n+2, \quad 2 n+4
$$

Their sum

$$
(2 n-4)+(2 n-2)+2 n+(2 n+2+(2 n+4)=10 n=200
$$

We calculate n :

$$
10 n=200, \quad n=200: 10=20
$$

Let's calculate five consecutive even numbers

$$
\begin{aligned}
& 2 n-4=2 * 20-4=36, \quad 2 n-2=2 * 20-2=38, \quad 2 n=2 * 20=40, \\
& 2 n+2=2 * 20+2=42, \quad 2 n+4=2 * 20+4=44 .
\end{aligned}
$$

Indeed

$$
36+38+40+42+44=200
$$

Question 9.3 The sum of four consecutive odd numbers is 160. Find these numbers.

## Solution:

The next four odd numbers are

$$
2 n-3, \quad 2 n-1, \quad 2 n+1 \quad 2 n+3
$$

Their sum

$$
(2 n-3)+(2 n-3)+(2 n+1)+(2 n+3)=8 n=160
$$

We calculate n :

$$
8 n=160, \quad \text { to } \quad n=160: 8=20
$$

Let's calculate four consecutive odd numbers

$$
\begin{aligned}
& 2 n-3=2 * 20-3=37, \quad 2 n-1=2 * 20-1=39 \\
& 2 n+1=2 * 20+1=41, \quad 2 n+3=2 * 20+3=43
\end{aligned}
$$

Indeed

$$
37+39+41+43=160
$$

Question 9.4 Between the digits of the number 18519 put the number 2 to get
(a) the largest number
(b) the least

Question 9.5 The sum of three consecutive even numbers is 36. find these numbers.
Question 9.6 The sum of four consecutive odd numbers is 180. Find these numbers.
Question 9.7 The sum of five consecutive even numbers equals 180. find these numbers.
Question 9.8 Calculate the sum

$$
S_{15}=1+2+3+4+5+6+7+8+9+10+11+12+13+14+15
$$

using one multiplication operation and one division operation.
Question 9.9 Calculate the sum

$$
S_{16}=2+4+6+8+10+12+14+16
$$

using one multiplication operation.
Question 9.10 Calculate the sum

$$
S_{21}=1+3+5+7+9+11+13+15+17+19+21
$$

using one multiplication operation.
Question 9.11 .
(a) Calculate the sum of the 20 words of the sequence

$$
3,6,9,12,15,18,21,24,27,30,33,36,39,42,45,48,51,54,57,60
$$

(b) Give the general formula for the sum of n-terms of the sequence

$$
3,6,9,12,15,18,21, \cdots, 3 n
$$

(c) Using this formula calculate the sum of 15 words of this sequence.

Question 9.12 Prove that an algebraic expression

$$
(a+1)(a+1)+4, \text { wewrite }
$$

is divisible by 4 for each even number $a$.

### 9.4 Binary system.

In the binary positional system the base $\rho=2$. We write binary numbers in the as the value of a polynomial for argument $x=2$.

$$
p_{n}(2)=a_{n-1} 2^{n-1}+a_{n-2} 2^{n-2}+\cdots+a_{1} 2+a_{0}=\left(a_{n-1} a_{n-2} \ldots a_{1} a_{0}\right)_{2}
$$

We see that there are only two digits 0,1 . Thus, the coefficients

$$
a_{0}, a_{1}, \cdots, a_{n-1}
$$

take the values 0 or 1 .
For example, a four-digit binary number

$$
x=\alpha_{3} \alpha_{2} \alpha_{1} \alpha_{0}=1010
$$

has
number of units $2^{0}=1, a_{0}=0$,
number of doubles $2^{1}$, $a_{1}=1$,
number of 2 's squares $2^{2}, a_{2}=0$
number of cubes of $2 \mathrm{~s} 2^{3}, a_{3}=1$.
Note that in the decimal system the base number is $\rho=10$. In the decimal system we write numbers using 10 digits

$$
0.1,2,3,4,5,6,7,8,9
$$

In contrast, in a binary system the base is the number $\rho=2$. In binary system there are two digits

$$
0,1
$$

which are also single-digit binary numbers.
We write two-digit binary numbers in general form

$$
a_{1} * 2+a_{0}=\left(a_{1} a_{0}\right)_{2}
$$

Example 9.15 Binary number $\quad x=(11)_{2}$, takes the form

$$
1 * 2+1=(11)_{2} .
$$

We write binary three-digit numbers in general form

$$
a_{2} * 2^{2}+a_{1} * 2^{1}+a_{0} * 2^{0}=\left(a_{2} a_{1} a_{0}\right)_{2}
$$

where the next powers of two

$$
2 * 2=2^{2}, \quad 2^{1}=2, \quad 2^{0}=1
$$

Example 9.16 For example, we write the binary number $x=(101)_{2}$ in general notation

$$
\begin{aligned}
a_{2} * 2^{2}+a_{1} * 2^{1}+a_{0} * 2^{0} & =\left(a_{2} a_{1} a_{0}\right)_{2} \\
& 1 * 2^{2}+0 * 2^{1}+1 * 2^{0}
\end{aligned}=(101)_{2}, ~ \$
$$

where the binary digit $a_{2}=1$ is the factor for $2^{2}$,
binary digit $a_{1}=0$ is the factor for 2 ,
binary digit of one $a_{0}=1$.
The value of this binary number

$$
(101)_{2}=1 * 2^{2}+0 * 2^{1}+1 * 2^{0}=5
$$

decimal is 5.

In general, we write n-digit numbers in positional binary as coefficients algebraic expression

$$
a_{n-1} 2^{n-1}+a_{n-2} 2^{n-2}+a_{n-3} 2^{n-3}+\cdots+a_{1} 2+a_{0}=\left(a_{n-1} a_{n-2} \ldots a_{1} a_{0}\right)_{2}
$$

where the successive powers of 2 are:

$$
\begin{aligned}
& 2^{1}=\underbrace{2}_{1} \\
& 2^{2}=\underbrace{2 * 2}_{2} \\
& 2^{3}=\underbrace{2 * 2 * 2}_{3} \\
& \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
& 2^{n-3}=\underbrace{2 * 2 * 2 * \ldots * \ldots}_{n-3} \\
& 2^{n-2}=\underbrace{2 * 2 * 2 * \ldots * 2}_{n-2} \\
& 2^{n-1}=\underbrace{2 * 2 * 2 * \ldots * 2}_{n-1}
\end{aligned}
$$

Example 9.17 Let $n=5$, then a five-digit binary number $x=(10101)_{2}$.
we write in the form of an arithmetic expression

$$
1 * 2^{4}+0 * 2^{3}+0 * 2^{2}+0 * 2^{1}+1 * 2^{0}=(10001)_{2}
$$

where the factor for $2^{4}$ is $a_{4}=1$,
the factor for $2^{3}$ is $a_{3}=0$,
the factor for $2^{2}$ is $a_{2}=0$,
wpfactor for $2^{1}$ is equal to $a_{1}=0$,
and the binary unity factor with $2^{0}$ is equal to $a_{0}=1$.

### 9.4.1 Decimal to binary conversion

Any decimal number can be converted to binary. This conversion is simple. Namely, we divide the decimal number by 2 and write the remainder. Then we divide the integer part of this division by 2 and write the remainder. We continue to divide integers by 2 , writing their remainders until after dividing by 2 we get an integer equal to 0 .
We get a binary number by writing the division remainder in order, starting with the last remainder and ending with the first remainder as the digit of binary unity. Let's see converting decimal numbers to binary for examples.

Example 9.18 Convert decimal $x=9$ to binary
We divide the decimal number $x=9$ by 2

$$
\begin{array}{lll}
\frac{9}{2}=4+\frac{1}{2} & \text { reminder } r_{0}=1, & 9=2 * 4+1 \\
\frac{4}{2}=2 & \text { reminder } r_{1}=0, & 4=2 * 2+0 \\
\frac{2}{2}=1 & \text { reminder } r_{2}=0, & 2=2 * 1+0 \\
\frac{1}{2}=0+\frac{1}{2} & \text { reminder } r_{3}=1, & 1=2 * 0+1
\end{array}
$$

By writing the remainder in the order from last to first we get a binary number

$$
\left(r_{3} r_{2} r_{1} r_{0}\right)_{2}=(1001)_{2}
$$

Let's repeat another division of 9 by 2 according to a different pattern used

$$
\begin{array}{lll}
\text { Liczba } x / 2 \\
====== & = & \text { Reszta zdzielenia przez } 2 \\
9 / 2=4 & 1 & ================ \\
4 / 2=2 & 0 \\
2 / 2=1 & 0 \\
1 / 2=0 & 1
\end{array}
$$

The result is a binary number by writing the remainder in order from last to first $(1001)_{2}$

Indeed

$$
(1001)_{2}=1 * 2^{3}+0 * 2^{2}+0 * 2^{1}+1 * 2^{0}=8+1=9
$$

Example 9.19 Convert decimal $x=15$ to binary
We divide the decimal number $x=15$ by 2

$$
\begin{array}{llll}
\frac{15}{2}=7+\frac{1}{2} & & \text { remainder } r_{0}=1 & \text { because }
\end{array} 15=2 * 7+1
$$

By writing the remainder from last to first, we get a binary number

$$
\left(r_{3} r_{2} r_{1} r_{0}\right)_{2}=(1111)_{2}
$$

Let us repeat successive divisions of 15 by 2 according to the used scheme

| Number $x / 2$ <br> $======$ | $=$remainder from divide by 2 <br> $===============$ <br> $15 / 2=7$ |
| :--- | :--- |
| $7 / 2=3$ | 1 |
| $3 / 2=1$ | 1 |
| $1 / 2=0$ | 1 |

As a result, we get a binary number by writing the remainder in order from last to first $(1111)_{2}$

Indeed

$$
(1111)_{2}=1 * 2^{3}+1 * 2^{2}+1 * 2^{1}+1 * 2^{0}=8+4+2+1=15 .
$$

### 9.4.2 General scheme for converting numbers from decimal to binary

As in the examples above, in the general scheme we divide the decimal number $x$ by 2 .

$$
\frac{x}{2}=k_{0}+\frac{r_{0}}{2}
$$

from where

$$
x=2 * k_{0}+r_{0}
$$

where $k_{0}$ is the total of displaystyle $E\left[\frac{x}{2}\right]$ and $r_{0}$ is the remainder of dividing $x$ by 2 Generally, we write

$$
\frac{k_{i}}{2}=k_{i+1}+\frac{r_{i+1}}{2}
$$

where

$$
k_{i}=2 * k_{i+1}+r_{i+1}, \quad i=0,1,2, \ldots, m
$$

for

$$
k_{i+1}=E\left[\frac{k_{i}}{2}\right] \quad i \quad r_{i+1} \quad \text { remainder with divide } k_{i} \text { by } 2
$$

### 9.4.3 Algorithm

Let's write the successive division in the following scheme

| Number $x$ | remainder |
| :--- | :--- |
| $==========$ | $====$ |
| $x / 2=k_{0}+r_{0} / 2$ | $r_{0}$ |
| $k_{0} / 2=k_{1}+r_{1} / 2$ | $r_{1}$ |
| $k_{1} / 2=k_{2}+r_{2} / 2$ | $r_{2}$ |
| $k_{2} / 2=k_{3}+r_{3} / 2$ | $\cdots$ |
| $\cdots$ | $r_{m-1}$ |
| $k_{m-2} / 2=k_{m-1}+r_{m-1} / 2$ | $r_{m}$ |
| $k_{m-1} / 2=0+r_{m} / 2$ | $r_{m}$ |

The result is a binary number by writing the remainder from last to first

$$
x=\left(y_{m} r_{m-1} r_{m-2} \ldots r_{1} r_{0}\right)_{2}
$$

### 9.4.4 Proof of algorithm

Note that the above-mentioned algorithm converts the decimal number $x$ to a binary number.
From this algorithm, we find

$$
\begin{array}{rl|l}
x & =2 k_{0}+r_{0} & k_{0}=2 k_{1}+r_{1} \\
& =2^{3} k_{2}+2^{2} r_{2}+2 r_{1}+r_{0} & k_{2}=2 k_{3}+r_{3} \\
& =2^{4} k_{3}+2^{3} r_{3}+2^{2} r_{2}+2 r_{1}+r_{0} & k_{3}=2 k_{4}+r_{4} \\
& \cdots \cdots \cdots \cdots \cdots+2^{2} & \cdots \cdots \cdots \\
& =2^{m-1} k_{m-2}+2^{m-2} r_{m-2}+\cdots+2^{2} r_{2}+2 r_{1}+r_{0} & k_{m-2}=2 k_{m-1}+r_{m-1} \\
& =2^{m} k_{m}+2^{m-1} r_{m-1}+\cdots+2^{2} r_{2}+2 r_{1}+r_{0} & k_{m-1}=2 k_{m}+r_{m} \\
& =2^{m} r_{m}+2^{m-1} r_{m-1}+\cdots+2^{2} r_{2}+2 r_{1}+r_{0} & k_{m}=r_{m} \\
& =\left(r_{m} r_{m-1} r_{m-2} \ldots r_{2} r_{1} r_{0}\right)_{2} &
\end{array}
$$

Let's apply the above algorithm converting the decimal number $x=256$ to binary.

| Number $x / 2$ <br> $======$ | remainder $z$ division by 2 <br> $==============$ |
| :--- | :--- |
| $256 / 2=128$ | 0 |
| $128 / 2=64$ | 0 |
| $64 / 2=32$ | 0 |
| $32 / 2=16$ | 0 |
| $16 / 2=8$ | 0 |
| $8 / 2=4$ | 0 |
| $4 / 2=2$ | 0 |
| $2 / 2=1$ | 0 |
| $1 / 2=0$ | 1 |

The result is a binary number by writing the remainder of the above table in order from last to first

$$
x=256=(100000000)_{2}
$$

Indeed
$(100.000 .000)_{2}=1 * 2^{8}+0 * 2^{7}+0 * 2^{6}+0 * 2^{5}+0 * 2^{4}+0 * 2^{3}+0 * 2^{2}+0 * 2^{1}+0 * 2^{0}=256$.

### 9.5 Binary arithmetic

Arithmetic operations in the binary system:
addition, subtraction, multiplication and division
are performed in a similar way as in the decimal system. We remind that in the decimal system, we perform arithmetic operations with the bese $\rho=10$ and decimal digits $0,1,2,3,4,5,6,7,8,9$.
In the binary system, we perform arithmetic operations when bases $\rho=2$ and when binary digits are 0,1

### 9.5.1 Binary Addition

The tablet of binary addition

| + | 0 | 1 |
| :---: | :---: | :---: |
| 0 | 0 | 1 |
| 1 | 1 | $(10)_{2}$ |

Binary sum

$$
\begin{aligned}
& 0+0=0 \\
& 0+1=1 \\
& 1+0=1 \\
& 1+1=(10)_{2}=1 * 2^{1}+0 * 2^{0}
\end{aligned}
$$

We explain binary addition by examples
Example 9.20 Do a binary addition of the decimals 5 and 3

We write the decimal number 5 in binary notation. (101) 2 and the decimal number 3 we write binary $(11)_{2}$.
Let us perform binary addition of $(101)_{2}+(11)_{2}$ using the binary addition table.

$$
\begin{array}{r}
101 \\
+\quad 11 \\
-- \\
1000
\end{array}
$$

Indeed

$$
5+3=(101)_{2}+(11)_{2}=(1000)_{2}=1 * 2^{3}+0 * 2^{2}+0 * 2^{1}+0 * 2^{0}=8
$$

### 9.5.2 Binary subtraction

The binary subtraction table

| - | 0 | 1 |
| :---: | :---: | :---: |
| 1 | 0 | -1 |
| 1 | 1 | 0 |

Binary substruction

$$
\begin{aligned}
& 0-0=0 \\
& 0-1=-1 \\
& 1-0=1 \\
& 1-1=0
\end{aligned}
$$

We explain binary subtraction by examples. Thus, we write the decimal number 5 in binary notation $(101)_{2}$ and the decimal number 3 we write binary $(11)_{2}$.
Let us perform binary substruction of $(101)_{2}-(11)_{2}$ using the binary addition table.
Example 9.21 Do a binary addition of the decimals 5 and 3
We do a written binary subtraction of $(101)_{2}-(11)_{2}$ using the binary subtraction table.
101
$-\quad 11$
---
10

Indeed

$$
5-3=(101)_{2}-(11)_{2}=(10)_{2}=1 * 2^{1}+0 * 2^{0}=2
$$

### 9.5.3 Binary Multiplication

Binary multiplication table

| ${ }^{*}$ | 0 | 1 |
| :---: | :---: | :---: |
| 0 | 0 | 0 |
| 1 | 0 | 1 |

Binary multiplication

$$
\begin{aligned}
& 0 * 0=0 \\
& 0 * 1=0 \\
& 1 * 0=0 \\
& 1 * 1=1
\end{aligned}
$$

We explain binary multiplication by examples
Example 9.22 Do a binary decimal multiplication of 5 and 3
We explain binary multiplication by examples. So, we write the decimal number 5 in binary notation $(101)_{2}$ and the decimal number 3 we write binary $(11)_{2}$.
Let us perform binary multiplication of $(101)_{2} *(11)_{2}$ using the binary addition table.
101

* 11
-     -         - 

101
101
1111
Verificaction:

$$
5 * 3=(101)_{2} *(11)_{2}=(1111)_{2}=1 * 2^{3}+1 * 2^{2}+1 * 2^{1}+1 * 2^{0}=15
$$

### 9.5.4 Binary division

We explain binary division by examples
Example 9.23 Do binary decimal division 15 divide by 3
We explain binary division by examples. Thus, we write the decimal number 5 in binary notation $(101)_{2}$ and the decimal number 3 we write binary $(11)_{2}$.
Let us perform binary division of $(101)_{2}:(11)_{2}$ using the binary addition table.
101

-     -         -             -                 - 

1111: 11
11
$=11$
11

-     -         - 

$=$

$$
5: 3=(101)_{2}:(11)_{2}=(101)_{2}=1 * 2^{2}+0 * 2^{1}+1 * 2^{0}=5
$$

### 9.6 Binary odd and even numbers

As in the decimal system, we recognize binaryeeven and odd numbers by the unity digit. Namely, if the unitary digit of a binary number is 0 then the binary number is even, otherwise, if the unitary digit of the binary number is 1 then the binary number is odd.

### 9.6.1 Binary even numbers

1. Even binary numbers have unity digit 0 .

For example, binary numbers

$$
10,110,1010,110110,111110110
$$

Since they have the unity digit 0 , so they are even.
2. Even binary numbers are divisible by binary $(10)-2$. The general form of even binary numbers is 1

$$
n=10 * k, \quad \text { for } k=0,10,100,110,1000, \ldots ;
$$

For example

$$
\begin{array}{ll}
k=0, & n=10 * 0=0 \\
k=1, & n=10 * 1=10 \\
k=10, & n=10 * 10=100 \\
\cdots & \cdots \cdots \cdots \\
k=1000, & n=1000 * 100=10000
\end{array}
$$

3. The sum, difference and product of even binary numbers is even binary For example:

$$
\begin{aligned}
a & =1000, \quad b=110 \\
a+b & =1000+110=1110 \\
a-b & =1000-110=10 \\
a * b & =1000 * 110=110000
\end{aligned}
$$

### 9.6.2 Binary odd numbers

1. Binary odd numbers have the units digit 1 .

For example, the following binary odd numbers are odd
$111,111,1011,110111,111110111$
2. Binary odd numbers have the general form

$$
n=(10)_{2} * k+1, \quad \text { or } n=(10)_{2} * k-1, \text { for } k=0,10,100,110,1000, \ldots
$$

For example

$$
\begin{array}{llll}
k=0, & n=10 * 0+1=1, & \text { or } & n=10 * 0-1=-1 \\
k=1, & n=10 * 1+1=11, & \text { or } & n=10 * 1-1=1 \\
k=10, & n=10 * 10+1=101, & \text { or } & n=10 * 10-1=11 \\
k=1000, & n=10 * 1000+1=10001, & \text { or } & n=10 * 1000-1=1111 \\
\cdots & \cdots \cdots & \cdots \cdots & \cdots \cdots
\end{array}
$$

[^5]3. The sum or difference of two odd binary numbers is even. For example
$$
101+11=1000, \quad 101-11=10
$$

Give another example.
4. The product of odd binary numbers is an odd number

For example:

$$
101 * 11=1111, \quad 111 * 101=100011
$$

Give another example.
5. Conversely, the sum of an odd binary and an even binary is an odd number.

For example

$$
101+110=1011
$$

Give another example
6. Similarly, the difference between an odd binary and an even binary is the odd. For example
7.

$$
111-100=11
$$

Give another example.

### 9.6.3 Examples

Question 9.13 The sum of two consecutive odd binary numbers is equal to (100000) ${ }_{2}$. Find these binary numbers.

## Solution:

Two consecutive odd binary numbers are

$$
(10)_{2} * n-1, \quad(10)_{2} * n+1
$$

Their sum ${ }^{2}$

$$
(10 * n-1)+(10 * n+1)=100 * n=100000
$$

We find n :

$$
100 * n=100000, \text { to } n=100000: 100=1000
$$

Let's calculate two consecutive odd numbers

$$
10 * n-1=10 * 1000-1=1111, \quad 10 * n+1=10 * 1000+1=1001
$$

Indeed, we Check in the binary system:

$$
(10 * n-1)+(10 * n+1)=10 * 1111+10 * 1001=11110+10010=100000
$$

Question 9.14 The sum of three consecutive even binary numbers is $(11000)_{2}$. Find these numbers.

[^6]
## Solution:

The consequtive three binary even numbers are:

$$
10 * n-10, \quad 10 * n, \quad 10 * n+10
$$

Their sum

$$
(10 * n-10)+(10 * n)+(10 * n+10)=110 * n=(11000)_{2}
$$

We calculate n:

$$
110 * n=11000, \quad n=11000: 110=100
$$

Let's calculate three consecutive binary even numbers

$$
\begin{gathered}
10 * n-10=10 * 100-10=110 \\
10 * n=10 * 100=1000 \\
10 * n+10=10 * 100+10=1010 \\
(110)_{2}+(1000)_{2}+(1110)_{2}+(1010)_{2}=(11000)_{2}
\end{gathered}
$$

Question 9.15 Calculate the sum of binary numbers

$$
S_{1010}=1+10+11+110+101+110+111+1000+1001+1010
$$

using only one binary multiplication operation and one binary division operation.

## Solution:

Let's write the sums in reverse order and add the equal sides as follows

$$
\begin{array}{ccc}
S_{1010} & = & 1+10+11+110+101+110+111+1000+1001+1010 \\
S_{10} & = & 1010+1001+1000+111+110+101+100+11+10+1 \\
--- & \cdots & --------------- \\
10 * S_{1010} & = & \underbrace{1011+1011+1011+1011+1011+1011+1011+1011+1011+1011}_{1010 \text { components of the sum }}
\end{array}
$$

Where do we calculate the sum of $S_{1010}$ using one binary multiplication operation and one binary division operation.

$$
\begin{aligned}
& (10)_{2} * S_{1010}=(1010)_{2} *(1011)_{2}=(1101110)_{2} \\
& S_{1010}=(1101110)_{2}:(10)_{2}=(11111)_{2}
\end{aligned}
$$

Check the solution in decimal.

### 9.6.4 Questions

Question 9.16 Convert decimals to binary using the conversion algorithm.
(a) $x=53$
(b) $x=1,025$

Check the obtained conversion results.

## Question 9.17 .

(a) Convert 513 and 25 decimal numbers to binary. Check the result
(b) Add binary numbers

$$
(1000000001)_{2}+(100001)_{2}
$$

## Question 9.18 .

(a) Convert 256 and 16 decimals to binary number. Check the result recalculation.
(b) Subtract binary numbers

$$
(100000000)_{2}-(1000)_{2}
$$

Check the result of the subtraction.

## Question 9.19 .

(a) Convert 129 and 3 decimals to binary number. Check the result of calculation.
(b) Multiply 129 and 3 in the binary system. Check the result of the multiplication.

## Question 9.20 .

(a) Convert 63 and 3 decimal number to binary number. Check the result. of calculation.
(b) Divide 63 by 3 in binary system. Check the division result.

Question 9.21 How many different three-digit binary numbers are there?
Question 9.22 Calculate the value of an arithmetic expression respecting the order of addition, subtraction, multiplication and division.

$$
(10)_{2} *(101)_{2}+(11)_{2} *(101)_{2}-(110)_{2}:(10)_{2}
$$

Question 9.23 Calculate the value of an arithmetic expression respecting the order of the arithmetic operations with parentheses.
(a)

$$
(100)_{2} *\left((10)_{2} *(101)_{2}+(11)_{2} *(101)_{2}\right)
$$

(b)

$$
(10)_{2} *\left((110)_{2}:(10)_{2}-(1000)_{2}:(100)_{2}\right)
$$

Question 9.24 The sum of five consecutive even binary numbers is $(100100)_{2}$. Find these numbers.

Question 9.25 Calculate the sum of even binary numbers

$$
\begin{aligned}
S_{10100} & =(10)_{2}+(100)_{2}+(110)_{2}+(1000)_{2}+(1010)_{2}+(1100)_{2}+ \\
& +(1110)_{2}+(10000)_{2}+(10010)_{2}+(10100)_{2}
\end{aligned}
$$

using only one binary multiplication operation.

### 9.7 Octal system.

In octal system the, base is $\rho=8$. Thus values of octal numbers in polynomial notation

$$
p(8)=a_{n-1} 8^{n-1}+a_{n-2} 8^{n-2}+\cdots+a_{1} 8^{1}+a_{0} 8^{0}=\left(a_{n-1} a_{n-2} \ldots a_{1} a_{0}\right)_{8}
$$

Octal digits are:

$$
0,1,2,3,4,5,6,7
$$

The digits of the octal system ${ }^{3}$

$$
\left(\alpha_{0}, \alpha_{1}, \cdots, \alpha_{n-1}\right)_{8}
$$

take values of $0,1,2,3,4,5,6,7$.
For example, octal $x=\left(\alpha_{3} \alpha_{2} \alpha_{1} \alpha_{0}\right)_{8}=(1257)_{8} \mathrm{ma}$
number of units $8^{0}=1, \alpha_{0}=7$,
number of eights $8^{1}, \alpha_{1}=5$,
number of eight squares $8^{2}, \alpha_{2}=2$
number of eight cubes $8^{3}, \alpha_{3}=1$.
Note that in the decimal system the base number is 10 and we write numbers using 10 digits

$$
0,1,2,3,4,5,6,7,8,9
$$

However, in the octal system the base is 8

$$
0,1,3,4,5,6,7
$$

and we write octal numbers using 8 octal digits For example, we write two-digit octal numbers in general form as follow

$$
a_{1} * 8+a_{0}=\left(a_{1} a_{0}\right)_{8}
$$

where the eighth digit is the factor $a_{1}$, the one digit is the factor $a_{0}$

Example 9.24 Octal number $\quad x=(65)_{8}$

$$
6 * 8+5 * 8^{0}=(65)_{8}
$$

Here the digit of eights is the factor $a_{1}=6$, the one digit is the factor $a_{0}=5$. The decimal value for this octal number is 53.
Indeed, let's take the decimal value of the octal number $(65)_{8}$

$$
(65)_{8}=6 * 8+5 * 1=53
$$

We write three-digit octal numbers in general form

$$
a_{2} * 8^{2}+a_{1} * 8^{1}+a_{0} * 8^{0}=\left(a_{2} a_{1} a_{0}\right)_{2}
$$

where the next powers of eight

$$
8 * 8=8^{2}, \quad 8^{1}=8, \quad 8^{0}=1
$$

[^7]Example 9.25 For example, octal $x=(256)_{8}$ in general notation we write

$$
\begin{aligned}
a_{2} * 8^{2}+a_{1} * 8^{1}+a_{0} * 8^{0} & =\left(a_{2} a_{1} a_{0}\right)_{2} \\
2 * 8^{2}+5 * 8^{1}+6 * 8^{0} & =(256)_{8}
\end{aligned}
$$

octal digit of units $a_{0}=6$.
The value of this number in the decimal system

$$
(256)_{8}=2 * 2^{2}+5 * 8^{1}+6 * 8^{0}=174
$$

In general, n -digit numbers in the positional octal system are written as coefficients of algebraic expression

$$
a_{n-1} 8^{n-1}+a_{n-2} 8^{n-2}+\cdots+a_{1} 8^{1}+a_{0} * 8^{0}=\left(a_{n-1} a_{n-2} \ldots a_{1} a_{0}\right)_{8}
$$

Example 9.26 Let $n=5$, then a four-digit octal number $x=(1024)_{8}$. we write in the form of an arithmetic expression

$$
1 * 8^{3}+0 * 8^{2}+2 * 8^{1}+4 * 8^{0}=(1024)_{8}
$$

where the factor for $8^{3}$ is $a_{3}=1$,
factor for $8^{2}$ is $a_{2}=0$,
factor for $8^{1}$ is $a_{1}=2$,
the one factor for $8^{0}$ is equal to $a_{0}=4$,
Let's calculate the decimal value of this number

$$
1 * 8^{3}+0 * 8^{2}+2 * 8^{1}+4 * 8^{0}=512+16+4=536
$$

### 9.7.1 Convert decimal numbers to octal numbers

Any decimal number can be converted to octal. As for the binary system, this conversion is simple. Namely, we divide the decimal by 8 and write the remainder. Then, we divide the the integer part of this division by 8 and write the reminder. We continue to divide the integers by 8 , writing down their remainders until we get an integer equal to 0 .
We get an octal number by writing the remainder of the division in order, starting with the last remainder and ending with the first remainder as the digit of octal unity. Let's see decimal to octal conversion for examples.

Example 9.27 Convert decimal $x=38$ to octal
We divide the decimal number $x=38$ by 8

$$
\begin{array}{llll}
\frac{38}{8}=4+\frac{6}{8} & \text { remainder } r_{0}=6 & \text { because } & 38=8 * 4+6 \\
\frac{4}{8}=0 & \text { remainder } r_{1}=4 & \text { because } & 4=0+4 * 1
\end{array}
$$

By writing the remainder in order from last to first, we get an octal number

$$
x=\left(r_{1} r_{0}\right)_{8}=(46)_{8}
$$

Let's repeat another division of 38 by 8 according to a different pattern used

| Number $x / 2$ | remainder division by 2 |
| :--- | :--- |
| $======$ | $=$ |
| $38 / 8=4$ | 6 |
| $4 / 8=0$ |  |
| $4 / 8=============$ |  |
|  |  |

The result is an octal number

$$
\begin{gathered}
x=(46)_{8} \\
x=(46)_{8}=4 * 8+6 * 8^{0}=8+1=38
\end{gathered}
$$

### 9.7.2 General scheme for converting numbers from decimal to octal

As in the examples above, in the general scheme, we divide the decimal number $x$ by 8 .

$$
\frac{x}{2}=k_{0}+\frac{r_{0}}{2}, \quad x=2 * k_{0}+r_{0}
$$

where $k_{0}$ is the integer part, $r_{0}$ is the remainder of dividing $x$ by 8

### 9.7.3 Algorithm

Let's write the above successive divisions by 8 in the following scheme

| Number $x$ | remainder |
| :--- | :--- |
| $==========$ | $====$ |
| $x / 8=k_{0}+r_{0} / 8$ | $r_{0}$ |
| $k_{0} / 8=k_{1}+r_{1} / 8$ | $r_{1}$ |
| $k_{1} / 8=k_{2}+r_{2} / 8$ | $r_{2}$ |
| $k_{2} / 8=k_{3}+r_{3} / 8$ | $r_{3}$ |
| $\cdots$ | $\cdots$ |
| $k_{m-2} / 8=k_{m-1}+r_{m-1} / 8$ | $r_{m-1}$ |
| $k_{m-1} / 8=0+r_{m} / 8$ | $r_{m}$ |

The result is an octal number by writing the remainder from last to first

$$
x=\left(r_{m} r_{m-1} r_{m-2} \ldots r_{1} r_{0}\right)_{8}
$$

### 9.7.4 Proof of algorithm

Note that the above-given algorithm converts the decimal number $x$ to the octal number.
From this algorithm, we find

$$
\begin{array}{rl|l}
x & =8 k_{0}+r_{0} & k_{0}=8 k_{1}+r_{1} \\
& =8^{3} k_{2}+8^{2} r_{2}+8 r_{1}+r_{0} & k_{2}=8 k_{3}+r_{3} \\
& =8^{4} k_{3}+8^{3} r_{3}+8^{2} r_{2}+8 r_{1}+r_{0} & k_{3}=8 k_{4}+r_{4} \\
& \cdots \cdots \cdots \cdots \cdots & \cdots \cdots \cdots \\
& =8^{m-1} k_{m-2}+8^{m-2} r_{m-2}+\cdots+8^{2} r_{2}+8 r_{1}+r_{0} & k_{m-2}=8 k_{m-1}+r_{m-1} \\
& =8^{m} k_{m}+8^{m-1} r_{m-1}+\cdots+8^{2} r_{2}+8 r_{1}+r_{0} & k_{m-1}=8 k_{m}+r_{m} \\
& =8^{m} r_{m}+8^{m-1} r_{m-1}+\cdots+8^{2} r_{2}+8 r_{1}+r_{0} & k_{m}=r_{m} \\
& =\left(r_{m} r_{m-1} r_{m-2} \ldots r_{2} r_{1} r_{0}\right)_{8} &
\end{array}
$$

End of the proof.
For example let's apply the above algorithm to convert the decimal number $x=256$ to octal.

$$
\begin{array}{lll}
\text { Number } x / 8 & \mid & \text { remainder division by } 8 \\
====== & = & ================ \\
256 / 8=32 & 0 \\
32 / 8=4 & 0 \\
4 / 8=0 & & 4
\end{array}
$$

The result is an octal number by writing the remainder of the above table in order from last to first

$$
\begin{gathered}
x=256=(400)_{8} \\
x=(400)_{8}=4 * 8^{2}+0 * 8^{1}+0 * 8^{0}=256
\end{gathered}
$$

### 9.8 Octal arithmetic

Arithmetic operations in the octal system, addition, subtraction, multiplication and division are performed in a similar way as in the decimal system. We remind that in the decimal system, the base $\rho=10$ by performing operations on decimal digits

$$
0,1,2,3,4,5,6,7,8,9
$$

Similarly, in octal system, the base is $\rho=8$ and operations are executed with octal digits

$$
0,1,2,3,4,5,6,7
$$

### 9.8.1 Octal addition

Let us write the octal addition table

|  | addition |  |  |  | octal |  |  |  |
| :---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| + | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 10 |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 | 10 | 11 |
| 3 | 3 | 4 | 5 | 6 | 7 | 10 | 11 | 12 |
| 4 | 4 | 5 | 6 | 7 | 10 | 11 | 12 | 13 |
| 5 | 5 | 6 | 7 | 10 | 11 | 12 | 13 | 14 |
| 6 | 6 | 7 | 10 | 11 | 10 | 13 | 14 | 15 |
| 7 | 7 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |

Example 9.28 Do octal addition of decimals 25 and 13
We write decimal 5 in octal notation $(31)_{8}$ and decimal 13 as octal $(15)_{8}$. Let us write octal addition $(31)_{8}+(13)_{8}$ using the octal addition table.

$$
\begin{gathered}
+\quad 15 \\
-- \\
46
\end{gathered}
$$

$$
(46)_{8}=4 * 8+6 * 8^{0}=38
$$

### 9.8.2 Octal subtraction

The octal subtraction table

|  | substeruction |  |  |  | octal |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :--- | :--- | :--- | :--- | :---: |
| - | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |  |
| 0 | 0 | -1 | -2 | -3 | -4 | -5 | -6 | -7 |  |
| 1 | 1 | 0 | -1 | -2 | -3 | -4 | -5 | -6 |  |
| 2 | 2 | 1 | 0 | -1 | -2 | -3 | -4 | -5 |  |
| 3 | 3 | 2 | 1 | 0 | -1 | -2 | -3 | -4 |  |
| 4 | 4 | 3 | 2 | 1 | 0 | -1 | -2 | -3 |  |
| 5 | 5 | 4 | 3 | 2 | 1 | 0 | -1 | -2 |  |
| 6 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | -1 |  |
| 7 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |  |

We explain octal subtraction by examples
Example 9.29 Do octal subtraction of the decimals 9 and 8
We write decimal 9 in octal notation $(11)_{8}$ and decimal 8 as octal $(10)_{8}$.
Let us write octal subtruction $(11)_{8}-(10)_{8}$ using the octal subtruction table.

$$
\begin{gathered}
11 \\
-\quad 10 \\
--- \\
1 \\
9-8=(11)_{8}-(10)_{8}=(1)_{8}=1 .
\end{gathered}
$$

### 9.8.3 Octal multiplication

The octal multiplication table

|  | multipliication |  |  |  | octal |  |  |
| :---: | :---: | :---: | :--- | :--- | :--- | :--- | :--- |
| $*$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 2 | 4 | 6 | 10 | 12 | 14 | 16 |
| 3 | 3 | 6 | 11 | 14 | 17 | 22 | 25 |
| 4 | 4 | 10 | 14 | 20 | 24 | 30 | 34 |
| 5 | 5 | 12 | 17 | 20 | 31 | 36 | 43 |
| 6 | 6 | 14 | 22 | 24 | 31 | 36 | 52 |
| 7 | 7 | 16 | 25 | 34 | 43 | 52 | 61 |

Example 9.30 Do the decimal multiplication of decimals9 and 15
We write decimal 9 in octal notation $(11)_{8}$ and decimal 8 as octal (10) 8 .
Let us write octal uultiplication $(11)_{8} *(10)_{8}$ using the octal multiplication table.

Multiplying decimals

$$
9 * 15=135
$$

Octal multiplication

$$
\begin{aligned}
& (11)_{8} *(17)_{8}=(207)_{8} \\
& (207)_{8}=2 * 8^{2}+0 * 8^{1}+7 * 8^{0}=2 * 64+7=135
\end{aligned}
$$

### 9.8.4 Octal division

We explain octal division by examples
Example 9.31 Do decimal octal division 45 divide by 3
We write decimal 45 in octal notation $(11)_{8}$ and decimal 3 as octal $(3)_{8}$.
Let us write octal division $(55)_{8}:(3)_{8}$.

$$
\begin{gathered}
17 \\
--- \\
55: 3 \\
-3 \\
-- \\
25 \\
25 \\
---- \\
= \\
45: 3=15 \\
(55)_{8}:(3)_{8}=(17)_{8}=1 * 8+7=15
\end{gathered}
$$

### 9.9 Octal numbers even and odd

As in the decimal system, even and odd octal numbers are identified by the unity digit. Namely, if the unity digit of an octal is 0 or 2 or 4 or 6 , then the octal is even, otherwise if the unity digit of an octal is 1 or 3 or 5 or 7 , then the octal number is odd.

### 9.9.1 Even octal numbers

1. Even octal numbers have unity digit

$$
0,2,4,6
$$

For example, octal even numbers

$$
0,2,4,6,10,12,14,16,20,22,24
$$

have the unity digit $0,24,6$.
2. Even octal numbers are divisible by the octal 2 . The general form of even octal numbers is: ${ }^{4}$

$$
n=2 * k, \quad \text { for } k=0,1,2,3,4,5,6,7,10,12,14, \ldots
$$

[^8]For example

$$
\begin{array}{ll}
k=0, & n=2 * 0=0 \\
k=1, & n=2 * 1=2 \\
k=2, & n=2 * 2=4 \\
k=3, & n=2 * 3=6 \\
k=4, & n=2 * 4=10 \\
k=5, & n=2 * 5=12,
\end{array}
$$

3. The sum, difference and product of even octal numbers is an even octal For example

$$
\begin{aligned}
& a=(12)_{8}, \quad b=(36)_{8} \\
& a+b=(12)_{8}+(36)_{8}=(50)_{8} \\
& a-b=(12)_{8}-(50)_{8}=-(24)_{8} \\
& a * b=(12)_{8} *(36)_{8}=(454)_{8}
\end{aligned}
$$

### 9.9.2 Odd octal numbers

1. Odd octals have unity digit 1 or 3 or 5 or 7 .

For example, octal numbers

$$
1,23,35,47,121,123,125,127
$$

have the unity digits $1,3,5,7,1,3,5,7$..
2. The general form of octal intege odd numbers is:

$$
n=(2)_{8} * k+1, \quad \text { or } n=(2)_{8} * k-1, \text { for } k=0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots ;
$$

For example

$$
\begin{array}{llll}
k=0, & n=2 * 0+1=1, & \text { or } & n=2 * 0-1=-1 \\
k=1, & n=2 * 1+1=3, & \text { or } & n=2 * 1-1=1 \\
k=2, & n=2 * 2+1=5, & \text { or } & n=2 * 2-1=3 \\
k=3, & n=2 * 3+1=7, & \text { or } & n=2 * 3-1=5 \\
k=4, & n=2 * 4+1=11, & \text { or } & n=2 * 4-1=7 \\
k=5, & n=2 * 5+1=13, & \text { or } & n=2 * 5-1=11 \\
k=6, & n=2 * 6+1=15, & \text { or } & n=2 * 6-1=13
\end{array}
$$

3. The sum or difference of two odd octal numbers is even.
4. For example

$$
(13)_{8}+(11)_{8}=(24)_{8}, \quad(13)_{8}-(11)_{8}=2
$$

Give another example.
5. The product of odd octal numbers is an odd number For example:

$$
(13)_{8} *(11)_{8}=(143)_{8}
$$

Give another example.
6. On the other hand, the sum of an odd octal and an even octal is an odd number.

For example
7.

$$
(26)_{8}+(15)_{8}=(43)_{8}
$$

Give another example
8. Similarly, the difference between an odd octal and an even octal is an odd number. For example
9.

$$
(26)_{8}-(15)_{8}=(11)_{8}
$$

Give another example.

### 9.9.3 Examples

Example 9.32 Convert decimals to octal numbers
(a) $x=100$
(b) $y=500$

## Solution (a):

We divide 100 decimal by 8 according to the scheme

$$
\begin{array}{lll}
\begin{array}{l}
\text { Number } x / 8 \\
======
\end{array} & = & \text { remainder division by } 8 \\
100 / 8=12 & & 4 \\
12 / 8=1 & & 4 \\
1 / 8=0 & & 1
\end{array}
$$

The octal of the decimal number $x=100$ is obtained by writing the remainder of this division from the last to the first

$$
\begin{gathered}
x=(144)_{8} \\
x=(144)_{8}=1 * 8^{2}+4 * 8+4=64+32+4=100
\end{gathered}
$$

## Solution (b):

We divide the decimal number 500 by 8 according to the scheme

$$
\begin{array}{lll}
\begin{array}{l}
\text { Number } x / 8 \\
======
\end{array} & =\begin{array}{l}
\text { remainder divisionby } 8 \\
================
\end{array} \\
500 / 8=62 & 4 \\
62 / 8=7 & 6 \\
7 / 8=0 & \mid & 7
\end{array}
$$

The writing of the decimal number $x=500$ is obtained by writing the remainder of this division from the last to the first

$$
\begin{gathered}
x=(764)_{8} \\
x=(764)_{8}=7 * 8^{2}+6 * 8+4=64+32+4=448+48+4=500
\end{gathered}
$$

Example 9.33 The sum of two consecutive odd octal numbers is (500) 8 . Find these binary numbers.

## Solution:

Two consecutive odd octal numbers are

$$
(2)_{8} * n-1, \quad(2)_{8} * n+1
$$

Their sum ${ }^{5}$

$$
(2 * n-1)+(2 * n+1)=4 * n=500
$$

We calculate n :

$$
4 * n=500, \quad \text { to } \quad n=500: 4=120
$$

Let us calculate two consecutive octal odd numbers

$$
\begin{aligned}
& (2)_{8} * n-1=(2)_{8} *(120)_{8}-1=(237)_{8} \\
& (2)_{8} * n+1=(2)_{8} *\left(120_{8}+1=(241)_{8}\right.
\end{aligned}
$$

Let us Check the resullt in the octal arithmetic

$$
\left.(2)_{8} * n-1\right)+\left((2)_{8} * n+1\right)=(237)_{8}+(241)_{8}=(500)_{8}
$$

Also, let us check the solution in decimal arithmetic.
Example 9.34 The sum of three consecutive even octal numbers is $(52)_{8}$. Find these numbers.

## Solution:

The next three even octal numbers are

$$
(2)_{8} * n-(2)_{8}, \quad(2)_{8} * n, \quad(2)_{8} * n+(2)_{8} .
$$

Their sum

$$
\left[(2)_{8} * n-(2)_{8}\right]+(2)_{8} * n+\left[(2)_{8} * n+(2)_{8}\right]=(6)_{8} * n=(52)_{8} .
$$

WE calculate n :

$$
(6)_{8} * n=(52)_{8}, \quad n=(52)_{8}:(6)_{8}=(7)_{8}
$$

Let's calculate three consecutive binary even numbers

$$
\begin{aligned}
& (2)_{8} * n-(2)_{8}=(2)_{8} *(7)_{8}-(2)_{8}=(14)_{8} \\
& (2)_{8} * n=(2)_{8} *(7)_{8}=(16)_{8} \\
& (2)_{8} * n+(2)_{8}=(2)_{8} *(7)_{8}+(2)_{8}=(20)_{8} \\
& \quad(14)_{8}+(16)_{8}+(20)_{8}=(52)_{8}
\end{aligned}
$$

[^9]Example 9.35 Calculate the sum of the octal numbers

$$
S_{20}=10+11+12+13+14+15+20
$$

using only one octal multiplication operation and one octal division operation.

## Solution.

Let's write the sum components in the reverse order and add the sum components side by side by doing octal addition on octal numbers as follows:

$$
\begin{gathered}
S_{20}=10+11+12+13+14+15+16+17+20 \\
S_{20}=20+17+16+15+14+13+12+11+10 \\
--- \\
2 * S_{20}=\underbrace{30+30+30+30+30+30+30+30+30}_{(11)_{8} \text { oktalnych skladnikow sumy }}
\end{gathered}
$$

Where do we calculate the sum of $S_{20}$ using one octal multiplication operation and one octal division operation

$$
\begin{aligned}
& (2)_{8} * S_{20}=(11)_{8} *(36)_{8}=(416)_{8} \\
& S_{20}=(416)_{8}:(2)_{8}=(207)_{8}
\end{aligned}
$$

Check the solution in decimal.

### 9.9.4 Questions

Question 9.26 Convert decimal numbers to octal numbers using the octal conversion algorithm.
(a) $x=53$
(b) $x=1025$

Check the obtained conversion results in the octal and decimal systems.

## Question 9.27 .

(a) Convert 513 and 25 decimals to octal numbers.
(b) Add the octal numbers

$$
(1003)_{8}+(10005)_{8}
$$

Check the result of the addition in the octal and decimal systems
Question 9.28 .
(a) Convert 256 and 16 decimals to octal numbers.
(b) Subtract octal numbers

$$
(10005)_{8}-(1003)_{8}
$$

Check the result of the subtraction in the octal and decimal systems.

## Question 9.29 .

(a) Convert 129 and 3 decimals to octal numbers.
(b) Multiply the numbers 129 and 3 in the octal system Check the result of multiplication in the octal and decimal systems.

Question 9.30 How many different two-digit octal numbers are there?
Question 9.31 Calculate the c value of the octal number arithmetic observing the order of addition, subtraction, multiplication and division

$$
(10)_{8} *(11)_{8}+(12)_{8} *(13)_{8}-(14)_{8}:(4)_{8}
$$

Question 9.32 Calculate the value of an arithmetic expression respecting the order of the arithmetic operations with parentheses.
(a)

$$
(2)_{8} *\left[(10)_{8} *(11)_{8}+(11)_{8} *(12)_{8}\right]
$$

(b)

$$
(3)_{8} *\left[(160)_{8}:(10)_{8}-(20)_{8}:(100)_{8}\right]
$$

Check the result of the octal computation in the decimal system.
Question 9.33 The sum of three consecutive even octal numbers is $(14)_{8}$. Find these numbers.

Question 9.34 Calculate the sum of the odd octal numbers

$$
S_{23}=(11)_{8}+(13)_{8}+(15)_{8}+(17)_{8}+(21)_{8}+(23)_{8}
$$

using only one octal multiplication operation

## Chapter 10

## Polynomials

By polynomials we understand the simplest class of functions with a very wide range of applications. The class of polynomials includes polynomials of degres $n=0,1,2,3, \ldots, m$, polynomials of single variable and more then one variables and polynomials of interpolation

In curiculum of primary and secondary schools polynomials appear in a very elementary form. In this chapter, polynomials are considered in the simplest form.

### 10.1 Monomials, binomials, and trinomials

A monomial is a sequence of numbers or a sequence of numbers and letters, or a sequence of only letters joined by the multiplication operation.
Let's list a few monomials

$$
\begin{array}{ll}
125 & 247, \quad \text { one number is amonomial } \\
2 * 5 * 7, & 3 * 4 * 5 * 6 * 7, \\
3 * a * b & a * b * c, \\
4 * * 5 * x * y * z, & 5 * a^{2} * b^{3} * c^{4}, \\
15 * x^{3} * y^{2} * z^{3} & 7 * 9 * a^{4} * b^{5} * x^{6} * y^{7} .
\end{array}
$$

Each monomial is a special arithmetic or algebraic expression because they contain numbers or letters that are only connected by a multiplication operation.
The binomial is the sum of two monomials.
For example

$$
a+b, \quad a-b, \quad a^{2}+b^{2}, \quad 3 x^{3}+5 y^{3} .
$$

Similarly, a trinomial is the sum of three monomials.
For example

$$
\begin{array}{lr}
a+b+c, & 2 * x_{3}+4 * y^{3}+5 * x * y, \\
a^{2}+2 * a * b+b^{2}, & x^{2}-2 * x * y+y^{2}
\end{array}
$$

### 10.2 Linear function.

A linear function, i.e. a polynomial of degree $n=1$ is a binomial of the special form:

$$
\begin{equation*}
w_{1}(x)=a x+b \tag{10.1}
\end{equation*}
$$

with the coefficients $a$ and $b$ and the variable $x$.
The binomial $w_{1}(x)=a x+b$ is called a linear function, because its graph is a straight
line. If the coefficient $a=0$ then the linear function is a constant whose graph is a straight line parallel to the $x$ axis. The linear function determines the relationship between the coordinates $x$ and $y$, which we write

$$
w_{1}(x)=a x+b, \quad \text { or } \quad y=a x+b
$$



Note that the straight line of the equation $y=x-1$ passes through the points $(0,-1),(1,0)$ and through the point $(2,1)$. We compute the values of this linear function below for the argument $x=0,1,2$

$$
w_{1}(0)=0-1=-1, \quad w_{1}(1)=1-1=0, \quad w_{1}(2)=2-1=1
$$

Let us observe that exactly one straight line passes through two different points. The equation of a line passing through two points with coordinates

$$
\left(x_{0}, y_{0}\right), \quad\left(x_{1}, y_{1}\right)
$$

we write as the following relation between $x$ and $y$ coordinate

$$
\begin{equation*}
y=\frac{x-x_{1}}{x_{0}-x_{1}} y_{0}+\frac{x-x_{0}}{x_{1}-x_{0}} y_{1} \tag{10.2}
\end{equation*}
$$

Indeed, when $x=x_{0}$ then $y=y_{0}$ or when $x=x_{1}$ then $y=y_{1}$.
It means that the points $\left(x_{0}, y_{0}\right),\left(x_{1}, y_{1}\right)$ lie on the straight line, because their coordinates satisfy the line equation.

Example 10.1 Write the equation of a straight line passing through two points $\left(x_{0}, y_{0}\right)=$ $(-1.0) \quad i\left(x_{1}, y_{1}\right)=(0.1)$. Which of the points $(1,1),(1,2)$ belongs to the straight line?

## Solution:

We write the equation of the straight line passing through the points

$$
\left(x_{0}, y_{0}\right)=(-1.0) \quad i \quad\left(x_{1}, y_{1}\right)=(0,1)
$$

by substituting to the formula (15.2) the coordinates

$$
y=\frac{x-x_{1}}{x_{0}-x_{1}} y_{0}+\frac{x-x_{0}}{x_{1}-x_{0}} y_{1}=\frac{x-0}{-1-0} * 0+\frac{x+1}{0+1} * 1=x+1
$$

Hence, we find the equation of a straight line passing through the points $(-1.0)$ and $(0.1)$

$$
y=x+1
$$

The point $(1,1)$ does not lie on the line $y=x+1$ because its coordinates do not satisfy the equation $1 \neq 1+1$. On the other hand, the point $(1,2)$ lies on the line $y=x+1$ because its coordinates satisfy the equality $2=1+1$ (see the figure below).


### 10.2.1 Position of straight lines on the plane.

Linear function

$$
y=a x+b
$$

determines the position of a straight line in the $(x, y)$ plane. That is, a point with the coordinates $(x, y)$ lies on a straight line, if

$$
y=a x+b
$$

Let us note that the graph of linear function $y=a x+b$ intersects the $x$ axis at $\left(\frac{-b}{a}, 0\right)$ and intersects the $y$ axis at $(0, b)$.
We say the number $x$ is proportional to the number $y$, if $y=a x$ or $\frac{y}{x}=a$. Thus, the proportionality of two quantities is expressed by a linear function, where coefficient $a \neq 0$ is the proportionality factor.

Note that graph of the line function $y=a x+b$

- intersects the $y$ axis, at the point $(0, b)$, if $x=0$, then

$$
y=a x+b=a * 0+b=b
$$

- intersects by $x$, at the point $\left(-\frac{b}{a}, 0\right)$, when $x=-\frac{b}{a}$, then

$$
y=a x+b=a\left(-\frac{b}{a}\right)+b=0 .
$$

- if $a=0$ then $y=b$ then the line is parallel to $x$ axis
- two lines with equations

$$
y=a_{1} x+b_{1}, \quad y=a_{2} x+b_{2}
$$

intersect at point $\left(x_{0}, y_{0}\right)$, if this point satisfies the equations of these lines

$$
y_{0}=a_{1} x_{0}+b_{1}, \quad i \quad y_{0}=a_{2} x_{0}+b_{2}
$$

- two lines are parallel if .zel $a_{1}=a_{2}$. Then the lines have no common point or overlap.

Example 10.2 Draw a graph of the position of two straight lines on $(x, y)$ plane

$$
y=x, \quad y=1-x
$$

Find the intersection points with the $x$ and $y$ axes and the point of intersection of the lines.
Solution. The line with the equation $y=x$ intersects the axis $x$ and the axis $y$ at the origin of the coordinate system (0.0), then $x=0$ and $y=0$.
Similarly, the line with the equation $y=1-x$ intersects by $x$, when $y=0$, i.e. $1-x=0$, for $x=1$, at the point (1.0). This line intersects the $y$ axis when $x=0$, then $y=1-0=1$ that is at $(0,1)$.
Two straight lines intersect at point $(x, y)$, when the coordinates of this point satisfy both equations, that is,

$$
y=x, \quad \text { and } \quad y=1-x
$$

Where from by the substitution $y=x$ to the second equation, we find

$$
x=1-x, \quad 2 x=1, \quad x=\frac{1}{2}, \quad y=\frac{1}{2} .
$$

Hence, the lines intersect at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$


Question 10.1 Draw a graph of the position of two lines with equations on the $(x, y)$ plane

$$
y=2 x-1, \quad y=1-2 x
$$

Find the intersection points with the $x$ and $y$ axes and the point of intersection of these lines.

Question 10.2 Write the equation of a straight line passing through two points $\left(x_{0}, y_{0}\right)=$ $(-1,-1) \quad i\left(x_{1}, y_{1}\right)=(1.1)$. Check which of the points $(0,1),(2,2)$ lies on a straight.

Question 10.3 At which points the line $y=-3 x+6$ intersects the coordinate axes. Calculate the value of this linear function for $x=1$. Check which point (0.3), (2.0) lies on the straight line.

### 10.3 Quadratic function

A quadratic function is given by the formula

$$
\begin{equation*}
w_{2}(x)=a x^{2}+b x+c, \quad \text { or } \quad y=a x^{2}+b x+c, \quad ; a \neq 0 \tag{10.3}
\end{equation*}
$$

In the case, when the coefficient $a=0$, the function is linear $y=b x+c$.
The domain of the quadratic function is the set of all real numbers $R$. On the other hand, the set of the values of a quadratic function depends on the coefficients $a, b, c$ and is not the whole set of real numbers.
The discriminant of a quadratic function. Expression

$$
\Delta=b^{2}-4 a c
$$

is called the discriminant of a quadratic function.

### 10.3.1 Quadratic equation

The quadratic function is zero at $x_{0}$, if $x_{0}$ is the solution to the quadratic equation

$$
a x^{2}+b x+c=0 .
$$

The roots of the quadratic equation are determined by complement method of the expresion

$$
a x^{2}+b x+c
$$

Namely, if we take the coefficient $a \neq 0$ in front of the parenthesis, we get

$$
a x^{2}+b x+c=a\left(x^{2}+\frac{b}{a} x+\frac{c}{a}\right) .
$$

Then, adding and subtracting the expression $\left(\frac{b}{2 a}\right)^{2}=\frac{b^{2}}{4 a^{2}}$, we write the canonical form of the quadratic expression

$$
a x^{2}+b x+c=a(\underbrace{x^{2}+\frac{b}{a} x+\frac{b^{2}}{4 a^{2}}}+\underbrace{\frac{c}{a}-\frac{b^{2}}{4 a^{2}}})=a[\underbrace{\left(x+\frac{b}{2 a}\right)^{2}}-\underbrace{\frac{b^{2}-4 a c}{4 a^{2}}}]
$$

This is how, we obtain the canonical form of the quadratic function, below
Canonical form of a quadratic function.

$$
y=a x^{2}+b x+c=a\left(x+\frac{b}{2 a}\right)^{2}-\frac{\Delta}{4 a}
$$

where the discriminant $\Delta=b^{2}-4 a c$.
Roots of the quadratic equation. From the canonical form of the quadratic function, we can easily find the roots of the quadratic equation. Namely, we write

$$
a x^{2}+b x+c=a\left(x+\frac{b}{2 a}\right)^{2}-\frac{\Delta}{4 a}=0 .
$$

For the discriminant $\Delta=b^{2}-4 a c \geq 0$, we can write the difference of squares as a product

$$
\left(x+\frac{b}{2 a}-\frac{\sqrt{\Delta}}{2 a}\right)\left(x+\frac{b}{2 a}+\frac{\sqrt{\Delta}}{2 a}\right)=0 .
$$

Hence, we find the formulas for the roots of the quadratic equation

$$
x_{1}+\frac{b}{2 a}-\frac{\sqrt{b^{2}-4 a c}}{2 a}=0, \quad \text { or } \quad x_{2}+\frac{b}{2 a}-\frac{\sqrt{b^{2}-4 a c}}{2 a}=0
$$

or

$$
x_{1}=\frac{-b-\sqrt{b^{2}-4 a c}}{2 a}, \quad \text { or } \quad x_{2}=\frac{-b+\sqrt{b^{2}-4 a c}}{2 a}
$$

Note that, when the discriminant $\Delta=0$, the quadratic function is a complte square

$$
a x^{2}+b x+c=a\left(x+\frac{b}{2 a}\right)^{2}
$$

Then, we obtain a double root from the above formulas

$$
a\left(x+\frac{b}{2 a}\right)^{2}=0, \quad x_{1}=x_{2}=\frac{-b}{2 a}
$$

### 10.3.2 Vieta formulas

Roots of the quadratic equation

$$
x^{2}+b x+c=0, \quad a \neq 0,
$$

satisfy the following Vieta formulas:
The sum and the product of the roots

$$
x_{1}+x_{2}=-\frac{b}{a}, \quad x_{1} * x_{2}=\frac{c}{a} .
$$

Indeed, we find

$$
x_{1}+x_{2}=\frac{-b-\sqrt{b^{2}-4 a c}}{2 a}+\frac{-b+\sqrt{b^{2}-4 a c}}{2 a}=-\frac{b}{a}
$$

Similarly, for the product of roots

$$
\begin{aligned}
x_{1} * x_{2} & =\left(\frac{-b-\sqrt{b^{2}-4 a c}}{2 a}\right) *\left(\frac{-b+\sqrt{b^{2}-4 a c}}{2 a}\right) \\
& =\left(-\frac{b}{2 a}\right)^{2}-\left(\frac{\sqrt{b^{2}-4 a c}}{2 a}\right)^{2} \\
& =\frac{b^{2}}{4 a^{2}}-\frac{b^{2}-4 a c}{4 a^{2}}=\frac{c}{a} .
\end{aligned}
$$

Example 10.3 Find a quadratic equation, when the sum of the roots is 3 and the product of the roots is 2.

Solution. Using Vieta's formulas, we write

$$
x_{1}+x_{2}=-\frac{b}{a}=3, \quad x_{1} * x_{2}=\frac{c}{a}=2 .
$$

Hence, we find

$$
b=-3 a, \quad c=a
$$

So that, we have a family of quadratic equations

$$
a x^{2}-3 a x+a=0
$$

with the parameter $a \neq 0$, whose sum of roots is 3 , and the product of roots is 2 .

Question 10.4 Find a quadratic equation, when the sum of the roots is 6 and the product of the roots is 5 .

Question 10.5 Find the value of the $\lambda$ parameter for which the equation has two different roots of the same sign
(i) $x * 2-2 x+\lambda-3=0$
(ii) $2 x * 2(\lambda+2) x-3 m-4=0$
(iii) $(\lambda+2) x^{2}-4 x+2 \lambda+6=0$

### 10.3.3 Factoring a quadratic function

If the discriminant $\Delta<0$ is negative, the quadratic equation has no real roots. Then the quadratic function does not decompose into linear factors.
When the discriminant $\Delta \geq 0$ is non-negative, the quadratic function decomposes into linear factors.
Indeed, we can represent the quadratic function as the difference of squares

$$
a x^{2}+b x+c=a\left[\left(x+\frac{b}{2 a}\right)^{2}-\left(\frac{\sqrt{\Delta}}{2 a}\right)^{2}\right]
$$

Applying the formula for the difference of squares, we obtain the decomposition of the quadratic function into linear factors

$$
a x^{2}+b x+c=a\left[\left(x+\frac{b}{2 a}-\frac{\sqrt{\Delta}}{2 a}\right)\left(x+\frac{b}{2 a}+\frac{\sqrt{\Delta}}{2 a}\right)\right]=a\left(x-x_{1}\right)\left(x-x_{2}\right)
$$

Position of the plot of a quadratic function on the plane. Location of the plot of a quadratic function on $9 x, y$ ) plane in coordinates $(x, y)$, we will determine in the following cases:
Case 1.


In the case (2)

$$
a<0, \quad \Delta>0, \quad \Delta=0, \quad \Delta<0
$$

we present the location of the plot of the square trinomial $y=a x^{2}+b x+c$


From the canonical form of the quadratic function, we conclude that

- square axis function ga minimum equal to $-\frac{\Delta}{4 a}$, if
$a>0$ is positive.
- The quadratic function reaches a maximum of $-\frac{\Delta}{4 a}$, if $a<0$ is negative.

Indeed, in the minimum or maximum point $\left(-\frac{b}{2 a},-\frac{\Delta}{4 a}\right.$ the square function reaches the minimum or maximum, because then in the canonical form

$$
y=a x^{2}+b x+c=a\left(x+\frac{b}{2 a}\right)^{2}-\frac{\Delta}{4 a},
$$

the expresion

$$
\left(x+\frac{b}{2 a}\right)^{2}=0
$$

for

$$
x=-\frac{b}{2 a}
$$

while the value of the function

$$
y=-\frac{\Delta}{4 a}
$$

Example 10.4 For a given quadratic function

$$
y=2 x^{2}-6 x+4
$$

perform the following operations:
(a) Find minimum values of the function
(b) Decompose functions into linear factors
(c) Find the minimum of the function
(d) Plot a graph of the function

Solution. The coefficients are: $a=2, b=-6, c=4$.
Let's calculate the discriminant of the equation

$$
\Delta=b^{2}-4 a c=6^{2}-424=36-32=4>0
$$

(a) Using the formulas, let's calculate the roots of the equation

$$
\begin{aligned}
& x_{1}=\frac{-b-\sqrt{\Delta}}{2 a}=\frac{6-\sqrt{4}}{4}=1 \\
& x_{2}=\frac{-b=\sqrt{\Delta}}{2 a}=\frac{6+\sqrt{4}}{4}=2
\end{aligned}
$$

(b) According to the formula, the quadratic function is decomposed into linear factors

$$
y=a\left(x-x_{1}\right)\left(x-x_{2}\right)=2(x-1)(x-2)
$$

(c) Since the discriminant $\Delta=4>0$ is positive, the quadratic function has the minimum

$$
-\frac{\Delta}{4 a}=-\frac{1}{2}
$$

in point $\left(\frac{-b}{2 a},-\frac{\Delta}{4 a}\right)=\left(\frac{3}{2},-\frac{1}{2}\right)$.
Points $(1,0)$ and $(2,0)$ in which are the roots of the square function and the minimum point $\left(\frac{3}{2} .-\frac{1}{2}\right)$ determine the position of its plot on the $(x, y)$ plane
(d) Plot of the function $y=2 x^{2}-6 x+4$


Quadratic function $y=2 x^{2}-6 x+4$.

### 10.3.4 Quadratic inequalities

The solution of the quadratic inequalities is obtained from the position of the plot of the quadratic function. Namely, we have the following cases:

1. for $a>0, \Delta>0$ square function $y=a x^{2}+b x+c>0$ is positive outside the roots: $x<x_{1}$ and $x>x_{2}$, and is negative $y=a x^{2}+b x+c<0$ beteen roots : $x_{1}<x<x_{2}$.
2. for $a<0, \Delta>0$ quadratic function $y=a x^{2}+b x+c>0$ is negative except for the roots: $x<x_{1}$ and $x>x_{2}$, while it is positive $y=a x^{2}+b x+c>0$ between the roots: $x_{1}<x<x_{2}$.

Example 10.5 Solve the following inequalities and find the maximum or minimum of the indicated functions:

$$
\begin{array}{ll}
\text { (1) } x^{2}+x+1>0, & y=x^{2}+x+1 . \\
\text { (2) }-2 x^{2}+2 x-1<0, & y=-2 x^{2}+2 x-1, \\
\text { (3) } x^{2}-5 x+6 \geq 0, & y=x^{2}-5 x+6, \\
\text { (4) }-2 x^{2}+x+1>0, & y=-2 x^{2}+x+1 .
\end{array}
$$

Solution, (1). We define the coefficients and the discriminant of the function

$$
y=x^{2}+x+1 .
$$

Coefficients:

$$
a=1, b=1, c=1 \text {. }
$$

and

$$
\Delta=b^{2}-4 a c=1^{2}-4 * 1 * 1=-3 .
$$

Since the coefficient $a=1>0$ is positive and the discriminant $\Delta=-3<0$ is negative, the inequality

$$
x^{2}+x+1>0,
$$

is true for $-\infty<x<\infty$.
Function

$$
y=x^{2}+x+1
$$

attains a minimum of $\frac{3}{4}$ at point $\left(-\frac{b}{2 a},-\frac{\Delta}{4 a}\right)=\left(-\frac{1}{2}, \frac{3}{4}\right)$.
Solution, (2). Determine the coefficients and the discriminant of the function

$$
y=-2 x^{2}+2 x-1 .
$$

Coefficients: $a=-2, b=2, c=-1$.
Discriminat: $\Delta=b^{2}-4 a c=2^{2}-4 *(-2) *(-1)=-4$.
Since the coefficient $a=-2<0$ is negative and the discriminant $\Delta=-4<0$ is negative, the inequality

$$
-2 x^{2}+2 x-1<0
$$

is true for $-\infty<x<\infty$.
The function $y=-2 x^{2}+2 x-1$ reaches a maximum of 1 at point $\left(-\frac{b}{2 a},-\frac{\Delta}{4 a}\right)=\left(\frac{1}{2}, 1\right)$
Solution, (3). We define the coefficients and the discriminant of the function

$$
y=x^{2}-5 x+6 .
$$

Coefficients: $a=1, b=-5, c=6$.
Distinctive: $\Delta=b^{2}-4 a c=(-5)^{2}-4 * 1 * 6=1$.
Since the discriminant $\Delta=1>0, \sqrt{1}=1$ is positive, the function has two different roots

$$
x_{1}=\frac{-b-\sqrt{\Delta}}{2 a}=\frac{5-1}{2}=2, \quad x_{2}=\frac{-b+\sqrt{\Delta}}{2 a}=\frac{5+1}{2}=3 .
$$

So inequality

$$
x^{2}-5 x+6 \geq 0
$$

is true beyond the square, that is, for $x<2$ and for $x>3$
The function $y=x^{2}-5 x+6$ reaches the minimums of $\frac{-\Delta}{4 a}=\frac{-1}{4}$ at the point $\left(\frac{-b}{2 a}, \frac{-\Delta}{4 a}\right)=\left(\frac{5}{2}, \frac{-1}{4}\right)$.
Solution, (4). We define the coefficients and the discriminant of the function

$$
y=-2 x^{2}+x+1
$$

Coefficients: $a=-2, b=1, c=1$.
Distinctive: $\Delta=b^{2}-4 a c=1^{2}-4 *(-2) * 1=9$.
Since the coefficient $a=-2<0$ the discriminant $\Delta=9>0, \quad \sqrt{9}=3$ is positive, the function

$$
y=-2 x^{2}+2 x-1
$$

it has two different elements

$$
x_{1}=\frac{-b-\sqrt{\Delta}}{2 a}=\frac{-1-3}{2 *(-2)}=1, \quad x_{2}=\frac{-b+\sqrt{\Delta}}{2 a}=\frac{-1+3}{2 *(-2)}=-\frac{1}{2} .
$$

So the inequality is true between roots, that is, for $-\frac{1}{2}<x<1$.
The function $y=-2 x^{2}+x+1$ has maximum equal to $\frac{-\Delta}{4 a}=\frac{9}{8}$ at the point $\left(\frac{-b}{2 a}, \frac{-\Delta}{4 a}\right)=\left(\frac{1}{4}, \frac{9}{8}\right)$.
Question 10.6 Solve the following inequalities and find the maximum or minimum of the indicated function:

$$
\begin{array}{ll}
\text { (1) } x^{2}-x+1>0, & y=x^{2}-x+1 . \\
(2)-3 x^{2}+6 x-3 \leq 0, & y=-3 x^{2}+6 x-3 . \\
(3) x^{2}-x-2 \geq 0, & y=x^{2}-x-2 . \\
(4)-4 x^{2}+3 x+1>0, & y=-4 x^{2}+3 x+1 .
\end{array}
$$

Question 10.7 For which values of the parameter $m$ the square function

$$
y=x^{2}+2 m x+m+1
$$

it is positive for all real values $x \in R$.
Example 10.6 For a square trnomial

$$
y=x^{3}-5 x+6
$$

(i) derive the canonical form of the trinomial
(ii) find its roots and find the minimum trinomial
(iii) draw the position of the trinomial on the Cartesian plane.

## Solution:

(i) The discriminant of a square trinom with coefficients $a=1, b-5, c=6$

$$
\Delta=b^{2}-4 a c=(-5)^{2}-41 * 6=25-24=1
$$

A simple transformation of this trinomial leads to a canonical form

$$
y=x^{2}-5 x+6=x^{2}-5 x+\left(\frac{-5}{2}\right)^{2}+6-\left(\frac{-5}{2}\right)^{2}=\left(x-\frac{5}{2}\right)^{2}-\frac{1}{4} .
$$

Where does the canonical post for this trinomial come from

$$
y=\left(x-\frac{5}{2}\right)^{2}-\frac{1}{4}
$$

(ii) Calculate the roots of the trinomial from the canonical form or directly from the formula. Namely, the canonical form is the difference of squares, which we factorize

$$
y=\left(x-\frac{5}{2}\right)^{2}-\frac{1}{4}=y=\left(x-\frac{5}{2}\right)^{2}-\left(\frac{1}{2}\right)^{2}=\left(x-\frac{5}{2}-\frac{1}{2}\right)\left(x-\frac{5}{2}+\frac{1}{2}\right) .
$$

Where do we calculate the roots of the quadratic equation from

$$
\begin{aligned}
& \left(x-\frac{5}{2}-\frac{1}{2}\right)\left(x-\frac{5}{2}+\frac{1}{2}\right)=0 \\
& \left(x-\frac{5}{2}-\frac{1}{2}\right)=0, \quad \text { or } \quad\left(x-\frac{5}{2}+\frac{1}{2}\right)=0 \\
& x_{1}=\frac{5}{2}+\frac{1}{2}=3, \quad x_{2}=\frac{5}{2}-\frac{1}{2}=2
\end{aligned}
$$

We can easily calculate the roots of a square trinomial by substituting into formulas

$$
x_{1}=-\frac{b}{2 a}-\frac{\sqrt{\Delta}}{2 a}=-\frac{-5}{2}-\frac{\sqrt{1}}{2}=2, \quad x_{2}=-\frac{b}{2 a}+\frac{\sqrt{\Delta}}{2 a}=-\frac{-5}{2}+\frac{\sqrt{1}}{2}=3 .
$$

We calculate the minimum square trinomial directly from the canonical form

$$
y=\left(x-\frac{5}{2}\right)^{2}-\frac{1}{4}
$$

Sure, the value of this trinomial is the smallest if the squared

$$
\left(x-\frac{5}{2}\right)^{2}=0
$$

For $x=\frac{5}{2}$, the value of $y=-\frac{1}{4}$. Thus, the minimum squared trinomial is $\frac{1}{4}$.


### 10.3.5 Examples

Example 10.7 Quadratic equation

$$
x^{2}-4 x+3=0
$$

has two real roots $x_{1}$ and $x_{2}$. Using the Viete formulas, calculate the values of algebraic expressions

$$
\left(x_{1}+x_{2}\right)^{2}, \quad x_{1}^{2}+x_{2}^{2}, \quad \frac{1}{x_{1}}+\frac{1}{x_{2}} .
$$

Solution: coefficients of the equation $a=1, b=-4, c=3$
From the Viete formula let's calculate the sum and the product of the primitives

$$
x_{1}+x_{2}=\frac{-b}{a}=\frac{-(-4)}{1}=4, \quad x_{1} * x_{2}=\frac{c}{a}=\frac{3}{1}=3
$$

Where do we calculate the values of algebraic expressions

$$
\left(x_{1}+x_{2}\right)^{2}=4^{2}=16, \quad x_{1}^{2}+x_{2}^{2}=\left(x_{1}+x_{2}\right)^{2}-2 x_{1} x_{2}=16-2 * 3=10
$$

or

$$
\frac{1}{x_{1}}+\frac{1}{x_{2}}=\frac{x_{1}+x_{2}}{x_{1} * x_{2}}=\frac{4}{3}
$$

Example 10.8 For which values of the parameter $m$ equation

$$
x^{2}-2 x+m=0
$$

has two different roots
Solution: The equation

$$
x^{2}-2 x+m=0
$$

has two different roots if the discriminant is positive

$$
\begin{aligned}
& \Delta=b^{2}-4 a c=(-2)^{2}-4 m>0 \\
& 4-4 m>0, \quad 4 m<4, \quad m<1
\end{aligned}
$$

Answer: The equation $x^{2}-2 x+m$ has two different roots for the parameter $-\infty<m<1$
Example 10.9 Find the coefficients $a, b, c$ of the quadratic equation

$$
a x^{2}+b x+c=0
$$

whose two real roots $x_{1}$ and $x_{2}$ satisfy Vieta's formulas

$$
x_{1}+x_{2}=7, \quad x_{1} * x_{2}=10
$$

Solution: Using Viete formulas

$$
x_{1}+x_{2}=\frac{-b}{a}=7, \quad x_{1} * x_{2}=\frac{c}{a}=10
$$

Solution: From Viete formulas

$$
b=-7 a, ; \quad c=10 a
$$

Hence the equation

$$
a x^{2}-7 a x+10 a=0, \quad \text { or } \quad a\left(x^{2}-7 x+10\right)=0
$$

satisfies the task condition for every $a \neq 0$.

Example 10.10 Find the coefficients $a, b, c$ of the quadratic equation

$$
a x^{2}+b x+c=0
$$

which has two real roots $x_{1}=3$ and $x_{2}=8$
Solution: Using Viete formulas

$$
x_{1}+x_{2}=3+8=11, \quad \frac{-b}{a}=11, \quad x_{1} * x_{2}=3 * 8=24, \quad \frac{c}{a}=24
$$

we find the following relationships

$$
b=-11 a, \quad c=24 a
$$

From where we get the equation

$$
a x^{2}-11 a x+24 a=0, \quad \text { or } \quad a\left(x^{2}-11 x+24\right)=0
$$

which has the roots $x_{1}=3, x_{2}=8$ for each $a \neq 0$.

### 10.3.6 Questions

Question 10.8 Find the roots of the equation
(i)) $x^{2}-3 x+6=0$,
(ii) $-2 x^{2}+9 x-10=0$,
(iii) $4 x^{2}-12 x+9=0$.

Question 10.9 For which value of $m$ the equation

$$
-x^{2}+4 x+m-4=0
$$

has two different roots
Question 10.10 For which values of the variable $x$ quadratic function

$$
y=x^{2}+4 x+3
$$

is positive.
Find smallest value of this function.
Question 10.11 For which values of the variable $x$ quadratic function

$$
y=-2 x^{2}+5 x+3
$$

is negative.
Find the greatest value of this function.
Question 10.12 For which values of the parameter $m$ quadratic function

$$
y=x^{2}+4 x+m^{2}
$$

is positive for all values of $x$.
Find smallest value of this function.
Question 10.13 For which values of the parameter $m$ quadratic function

$$
y=-x^{2}+3 x-m
$$

is negative for all values of $x$.
Find the greatest value of this function.
Question 10.14 Find a quadratic equation with the sum of the roots equal to 6 and the product of the roots equal to 5 .

### 10.4 Polynomials of degree $n$

Polynomials have a simple structure and constitute an important class of functions in applications. One of the important application is possibility to approximte a funcgtion with any precision.
A polynomial of degree $n$ of the variable $x$ is an algebraic expression of the following form:

$$
p_{n}(x)=a_{n} x^{n}+a_{n-1} x^{n-1}+a_{n-2} x^{n-2}+\cdots+a_{1} x+a_{0}, \quad a_{n} \neq 0
$$

If $a_{n}=0$ then the polynomial is of lower degree than. $\mathrm{z} n$

### 10.4.1 Examples of polynomials

Polynomial of degree $n=0$

$$
p_{0}(x)=a_{0} \quad \text { for all real values } \quad x \in(-\infty, \infty)
$$

For example, a polynomial of degree $n=0$

$$
p_{0}(x)=8 \quad \text { for all real values } \quad x \in(-\infty, \infty)
$$

has a constant value, $a_{0}=8$ for all xrealvalues.
Polynomial of degree $n=1$ of the variable $x$. It is the linear function

$$
p_{1}(x)=a_{1} x+a_{0} \quad \text { for all real values } \quad x(-\infty, \infty)
$$

For example, a polynomial of degree $n=1$

$$
p_{1}(x)=5 x+7 \quad \text { for } \quad x \in(-\infty, \infty)
$$

has the coefficients $a_{1}=5, a_{0}=7$.
Polynomial of degree $n=2$ of the variable $x$. It is the quadratic function

$$
p_{2}(x)=a_{2} x+a_{1} x^{2}+a_{0} \quad \text { for } x \in(-\infty, \infty)
$$

For example, a polynomial of degree $n=2$

$$
p_{2}(x)=3 x^{2}+4 x+5 \quad \text { for } \quad x \in(-\infty, \infty)
$$

has the coefficients $a_{2}=3, a_{1}=4, a_{0}=5$.
Polynomial of degree $n=3$ of the variable $x$. It is cubic polynomial

$$
p_{3}(x)=a_{3} x^{3}+a_{2} x^{2}+a_{1} x+a_{0} \quad \text { for } \quad x \in(-\infty, \infty)
$$

For example, a cubic polynomial

$$
p_{3}(x)=2 x^{3}+3 x^{3}+4 x+5, \quad \text { for } \quad x \in(-\infty, \infty)
$$

has the coefficients $a_{3}=2, a_{2}=3, a_{1}=4, a_{0}=5$.
Similarly, a polynomial of degree $n=5$ of the variable $z$, For example, a polynomial of degree $n=5$

$$
p_{5}(z)=2 z^{5}-7 z^{4}+5 z^{2}+2, \quad \text { for } \quad z \in(-\infty, \infty)
$$

has the coefficients $a_{5}=2, a_{4}=-7, a_{3}=0, a_{2}=5, a_{1}=0, a_{0}=2$.

### 10.4.2 Arithmetic operations on polynomials.

Let us state following theorem.
Theorem 10.1 The set of polynomials of degree not greater than $n$ is closed due to addition and subtraction operations.

Indeed, consider the following two polynomials

$$
p_{n}(x)=a_{n} x^{n}+a_{n-1} x^{n-1}+\cdots+a_{1} x+a_{0}, \quad q_{n}(x)=b_{n} x^{n}+b_{n-1} x^{n-1}+\cdots+b_{1} x+b_{0},
$$

We find the sum or difference of these polynomials by grouping words at the same power

$$
p_{n}(x) \pm q_{n}(x)=\left(a_{n} \pm b_{n}\right) x^{n}+\left(a_{n-1} \pm b_{n-1}\right) x^{n-1}+\cdots+\left(a_{1} \pm b_{1}\right) x+\left(a_{0} \pm b_{0} .\right)
$$

We note that as a result we get a polynomial of degree not more than $n$ with coefficients

$$
a_{n} \pm b_{n}, a_{n-1} \pm b_{n-1}, \ldots, a_{1} \pm b_{a}, a_{0} \pm b_{0}
$$

Thus, the sum or difference of degree polynomials of at most $n$ is a polynomial of degree of at most $n$. This means that the set of degree polynomials at most $n$ is closed to addition and subtraction operations of polynomials degree at most $n$.

Example 10.11 Add the following polynomials

$$
p_{4}(x)=3 x^{4}-2 x^{3}+x+5, \quad q_{3}(x)=2 x^{3}+5 x^{2}+2 x+1,
$$

By addition, we find the polynomial

$$
\begin{aligned}
r_{4}(x) & =(3+0) x^{4}+(-2+2) x^{3}+(0+5) x^{2}+(1+2) x+(5+1) \\
& =3 x^{4}+5 x^{2}+3 x+6
\end{aligned}
$$

Hence, we get polynomial of degree $n=4$ with coefficients $a_{4}=3, a_{3}=0, a_{2}=5, a_{1}=$ 3 , $a_{0}=6$.

### 10.4.3 Dividing the polynomial $p_{n}(x)$ by the binomial $x-x_{0}$

The polynomial $p_{n}(x)$ of degree $n$ is divided by the binomial $x-x_{0}$ of degree $n=1$ according to the division scheme given in the following examples:

Example 10.12 Let us execute division $x^{3}-1$ by $x-1$

$$
\begin{aligned}
& \left(x^{3}-1\right):(x-1)=x^{2}+x+1 \\
& x^{3}-x^{2} \\
& ---- \\
& x^{2}-1 \\
& x^{2}-x \\
& ---- \\
& x-1 \\
& x-1 \\
& ---- \\
& 0
\end{aligned}
$$

Thus, the polynomial $x^{3}-1$ is divissoble by the binomial $x-1$ and the result is the trinomial $x^{2}+x+1$.
We check the division by performing the operation inverse to division i.e. multiplication

$$
(x-1)\left(x^{2}+x+1\right)=x^{3}+x^{2}-x^{2}-1=x^{3}-1
$$

Indeed, by multiplying the divisor of $x-1$ by the result of dividing $x^{2}+x+1$, we get the divisor $x^{3}-1$.

Example 10.13 Let us perform the following diivision

$$
\begin{aligned}
& \left(x^{4}-x^{3}-x^{2}-x-2\right):(x-2)=x^{3}+x^{2}+x+1 \\
& x^{4}-2 x^{3} \\
& ---- \\
& x^{3}-x^{2} \\
& x^{3}-2 x^{2} \\
& ---- \\
& x^{2}-x \\
& x^{2}-2 x \\
& ---- \\
& x-2 \\
& x-2 \\
& ---- \\
& 0
\end{aligned}
$$

So, the polynomial $x^{4}-x^{3}-x^{2}-x-2$ is divissible by the binomial $x-2$ and the division result is $x^{3}+x^{2}+x+1$.
We check the result of divition by myltiplication
$(x-2)\left(x^{3}+x^{2}+x+1\right)=x^{4}+x^{3}+x^{2}+x-2 x^{3}-2 x^{2}-2 x-2=x^{4}-x^{3}-x^{2}-x-2$.
Question 10.15 Execute the division according to the above pattern:

$$
\left(x^{4}-1\right):(x-1)
$$

### 10.4.4 Dividing $p_{n}(x)$ by the binomial $x-x_{0}$ with remainder.

In the above examples, we divided the 3 rd and 4 th degree polynomial by the binomial $x-x_{0}$ with no remainder, i.e. the remainder $r=0$. However, this is not always the case.
Let us divide the polynomial

$$
p_{n}(x)=a_{n} x^{n}+a_{n-1} x^{n-1}+\cdots+a_{1} x+a_{0}, \quad n \geq 1
$$

by binomial $x-x_{0}$ with the remainder $r$. Just like when dividing integers, we write

$$
\frac{p_{n}(x)}{x-x_{0}}=q_{n-1}(x)+\frac{r}{x-x_{0}}, \quad n \geq 1
$$

where $q_{n-1}(x)$ is a polynomial of degree $n-1$ and $r$ is the remainder of the division. So the , polynomial $p_{n}(x)$ we can write in the form

$$
p_{n}(x)=q_{n-1}(x)\left(x-x_{0}\right)+r
$$

From the above equality, the remainder, we fin formula for reminder $r=p_{n}\left(x_{0}\right)$.

Example 10.14 Let us execute division

$$
\begin{aligned}
& \quad\left(2 x^{4}+3 x^{3}-4 x^{2}+5 x+6\right):(x-3) \\
& \left(2 x^{4}+3 x^{3}-4 x^{2}+5 x+6\right):(x-3)=2 x^{3}+9 x^{2}+23 x+74 \\
& 2 x^{4}-6 x^{3} \\
& ------- \\
& 9 x^{3}-4 x^{2} \\
& 9 x^{3}-27 x^{2} \\
& ---- \\
& 23 x^{2}+5 x \\
& 23 x^{2}-69 x \\
& ------ \\
& 74 x+6 \\
& 74 x-222 \\
& ----- \\
& 226
\end{aligned}
$$

The polynomial $p_{4}(x)=2 x^{4}+3 x^{3}-4 x^{2}+5 x+6$ divided by the binomial $x-3$ returns $q_{3}(x)=2 x^{3}+9 x^{2}+23 x+74$ with the remainder of $r=226$.
We write

$$
\frac{p_{4}(x)}{x-3}=\left(2 x^{3}+9 x^{2}+23 x+74\right)+\frac{226}{x-3}
$$

or

$$
p_{4}(x)=2 x^{4}+3 x^{3}-4 x^{2}+5 x+6=\left(2 x^{3}+9 x^{2}+23 x+74\right)(x-3)+226 .
$$

Hence, the remainder of the division $r=p_{4}(3)=226$.

### 10.4.5 Roots of polynomials. Bezout's Theorem

The zeros of a linear or quadratic function, i.e. of polynomials of first or second degree, can be easily found using the known formulas given in the previous paragraphs. Formulas for the roots of polynomials of the third and fourth degree are also known. But, for polynomials of degree greater or equal than 5 formulas for roots do not exis.
Namely, it is known, that if a polynomial with integer coefficients has integer roots, then these roots are divisors of its coefficient $a_{0}$. This criterion only applies to polynomials with integer coefficients. The justification for the criterion is simple. Indeed, let the integer number $x_{0} \neq 0$ be a root of the polynomial $p_{n}(x)$ of degree $n$ with integer coefficients. We will show that $x_{0}$ is a divisor of the coefficient $a_{0}$.
Let us write the obvious equality, when $p_{n}\left(x_{0}\right)=0$.

$$
\begin{equation*}
\frac{p_{n}\left(x_{0}\right)}{x_{0}}=\underbrace{a_{n} x_{0}^{n-1}+a_{n-1} x_{0}^{n-2}+\cdots+a_{1}}+\frac{a_{0}}{x_{0}}=0 \tag{10.4}
\end{equation*}
$$

The underlined expression $a_{n} x_{0}^{n-1}+a_{n-1} x_{0}^{n-2}+\cdots+a_{1}$ is an integer as the sum of the products of integers. From equality (10.4) it follows that the quotient $\frac{a_{0}}{x_{0}}$ is also a number integer, since the sum is zero. So, the root of $x_{0}$ is a divisor of the intercept $a_{0}$.

Example 10.15 Find the integer roots of a polynomial

$$
p_{3}(x)=x^{3}-x^{2}+x-6
$$

Solution. The zeros of the cubic polynomial $p_{3}(x)=x^{3}-x^{2}+x-6=0$ are among divisors 2 or 3 of the coefficient $a_{0}=-6$.
We check, if $x_{0}=2$ is zero of this polynomial

$$
p_{3}(2)=2^{3}-2^{2}+2-6=8-4+2-6=0
$$

Hence $x_{0}=2$ is zero of the polynomial $p_{4}(x)$.
Next we check, if $x_{0}=3$ is zero of this polynomial

$$
p_{3}(2)=3^{3}-3^{2}+3-6=27-9+3-6=12 \neq 0
$$

So, $x_{0}=3$ is not zero of this polynomial.
Note that there are polynomials for which none of the divisors of the coefficient $a_{0}$ is zero. For example, the quadratic polynomial

$$
p_{2}(x)=x^{2}+2 x+8
$$

has no real zeros, because the discriminant $\Delta=-28$ is negative.
In order to find zeros of a polynomial with integer coefficient, we apply Bezout theorem.
Theorem 10.2 The number $x_{0}$ is a zero of the polynomial

$$
p_{n}(x)=a_{n} x^{n}+a_{n-1} x^{n-1}+\cdots+a_{1} x+a_{0}, \quad n \geq 1
$$

if and only if the polynomial $p_{n}(x)$ is divisible by the binomial $x-x_{0}$.
Proof. Note that Bezout's theorem is a necessary and sufficient condition for the number $x_{0} \in R$ to be a zero of the polynomial.
The necessary condition means:
If the polynomial $p_{n}(x)$ is divisible by the binomial $x-x_{0}$, then the number $x_{0}$ is the zero of the polynomial, i.e. $p_{n}\left(x_{0}\right)=0$ and the remainder $r=0$.
Let the polynomial $p_{n}(x)$ be divisible by the binomial $x-x_{0}$ with no remainder. Then the polynomial has the form

$$
p_{n}(x)=\left(x-x_{0}\right) q_{n-1}(x)
$$

where $q_{n-1}(x)$ is a polynomial of degree $n-1$, at most.
From where $x=x_{0}$ equals $p_{n}\left(x_{0}\right)=0$ and therefore $x_{0}$ is the root of this polynomial.
The sufficient condition means:
If the number $x_{0} \in R$ is the root of the polynomial $p_{n}(x)$, then this polynomial is divisible by the binomial $x-x_{0}$, and reminder $r=0$.
It is known that dividing the polynomial $p_{n}(x)$ by the binomial $x-x_{0}$, we get the equality

$$
p_{n}(x)=q_{n-1}(x)\left(x-x_{0}\right)+r
$$

where $q_{n-1}(x)$ is a polynomial of degree $n-1$.
Since .z $x_{0}$ is zero of this polynomial, that is, $p_{n}\left(x_{0}\right)=0$ and $p_{n}\left(x_{0}\right)=r$. So the remainder of $r=0$. Then the form of the polynomial follows from the above equalit

$$
p_{n}(x)=q_{n-1}(x)\left(x-x_{0}\right)
$$

in which there is a factor $x-x_{0}$ and therefore the polynomial $p_{n}(x)$ is divisible by the binomial $x-x_{0}$ with the remainder $r=0$.

### 10.4.6 Decomposition of polynomials

From Bezout's theorem follws conclusion:
Conclusion. If real numbers $x_{1}, x_{2}, \ldots, x_{k}, k \leq n$ are zeros of the polynomial

$$
p_{n}(x)=a_{n} x^{n}+a_{n-1} x^{n-1}+\cdots+a_{1} x+a_{0}, \quad n \geq 1
$$

then this polynomial can be written as a product

$$
\begin{equation*}
p_{n}(x)=\left(x-x_{1}\right)\left(x-x_{2}\right) \cdots\left(x-x_{k}\right) q_{n-k}(x) \tag{10.5}
\end{equation*}
$$

of $n-k$ linear factors $\left(x-x_{i}\right), \quad i=1,2, \ldots, k$, and polynomial $q_{n-k}(x)$ of degree $n-k$. Indeed, for $k=1$, Bezout's theorem directly results in a product

$$
p_{n}(x)=\left(x-x_{1}\right) q_{n-1}(x)
$$

Applying Bezout's theorem again to the polynomial $q_{n-1}(x)$ for zero $x_{2}$, we get the distribution

$$
p_{n}(x)=\left(x-x_{1}\right)\left(x-x_{2}\right) q_{n-2}(x)
$$

By repeating the application of Bezout's theorem for the next zeros of the polynomial $p_{n}(x)$, we get the decomposition (10.5) of the polynomial into linear factors in the case when all zoros $x_{1}, \ldots, x_{n}$ are real numbers.

$$
p_{n}(x)=a\left(x-x_{1}\right)\left(x-x_{2}\right)\left(x-x_{n-3}\right) * \ldots *\left(x-x_{n}\right)
$$

where $q_{0}(x)=a=$ constant .

Otherwise, there are quadratic factors $q_{k+1}(x)=a x^{2}+b x+c$ with negative discriminant $\Delta=b^{2}-4 a * c<0$ in the factorization of the polynomial

$$
p_{n}(x)=a\left(x-x_{1}\right)\left(x-x_{2}\right)\left(x-x_{n-3}\right) * \ldots *\left(x-x_{k}\right) * \ldots * q_{k+1}(x)
$$

Now we will formulate the fundamental theorem on decomposition of the polynomial irreducible factors:

Theorem 10.3 Every polynomial

$$
p_{n}(x)=x^{n}+a_{n-1} x^{n-1}+\cdots+a_{1} x+a_{0}, \quad n \geq 1,
$$

can be decomposed into linear factors $x-x_{0}$ or quadratic factors $x^{2}+a_{1} x+a_{0}$ with the discriminant $\Delta=a_{1}^{2}-4 a_{0}<0$ negative. This decomposition is unequivocal.

In factorization of a polynomial the following operations are applied

1. The decomposion of a quadratic trinomial $a x^{2}+b x+c$
2. Extracting a common factor in front of the parenthesis
3. By grouping like expresions
4. By using simplification formulas of multiplication
5. By finding zeros of a polynomial with integer coefficients.

Example 10.16 Let us factorize the quadratic polynomial

$$
p_{2}(x)=a x^{2}+b x+c
$$

Solution. The quadratic polynomial is factorized depending on the sign of the value of the discriminant $\Delta=b^{2}-4 a c$. Namely, if the discriminant $\Delta \geq 0$ is non-negative, then this trinomial has two real roots and decomposes into linear factors

$$
a x^{2}+b x+c=a\left(x-x_{1}\right)\left(x-x_{2}\right)
$$

This case also includes the double root when $\Delta=0$ and $x_{1}=x_{2}$.
If the discriminant $\Delta<0$ is negative then the trinomial $a x^{2}+b x+c$ is not factorized and then the expression $a x^{2}+b x+c$ is the factor.

Example 10.17 Let us factorize the following polynomial by grouping similar expressions and extracting a common factor

$$
p_{3}(x)=x^{3}-2 x^{2}-4 x+8
$$

Solution. In this case, we group simlar terms. Then, we take $x^{2}$ and 4 in front of the bracket as it is done below

$$
\begin{aligned}
p_{3}(x)=x^{3}-2 x^{2}-4 x+8 & =x^{2}(x-2)-4(x-2) \\
& =(x-2)\left(x^{2}-4\right)
\end{aligned}
$$

Further, using the formula for the difference of squares $x^{2}-4=(x-2)(x+2)$ we get the factorization of this polynomial

$$
\begin{aligned}
p_{3}(x)=x^{3}-2 x^{2}-4 x+8 & =x^{2}(x-2)-4(x-2)=(x-2)\left(x^{2}-4\right) \\
& =(x-2)(x-2)(x+2)=(x-2)^{2}(x+2)
\end{aligned}
$$

Example 10.18 Let us factorize the cubic polynomial

$$
p_{3}(x)=x^{3}+5 x^{2}+2 x+10
$$

Solution. In this case, if we put $x^{2}$ and 5 in front of the bracket, then we get decomposition of $p_{3}(x)$

$$
\begin{aligned}
p_{3}(x)=x^{3}+5 x^{2}+2 x+10 & =x^{2}(x+5)+2(x+10) \\
& =(x+5)\left(x^{2}+2\right)
\end{aligned}
$$

Since the quadratic expression $x^{2}+2>0$ is positive for all real $x$, then, we get decomposition with quadratic term $x^{2}+2$

$$
\begin{aligned}
p_{3}(x)=x^{3}+5 x^{2}+2 x-10 & =x^{2}(x+5)+2(x+10) \\
& =(x+5)\left(x^{2}+2\right)
\end{aligned}
$$

Example 10.19 Let us factorize the following polynomial

$$
p_{4}(x)=x^{4}-4 x^{3}-x^{2}+16 x-12
$$

Solution. In this case, we look for the integer zeros of the polynomial with integer coefficients among the divisors $-2,-1,1,2,3,4,6$ of the coefficient $a_{0}=-12$.

1. We check, if the divisor of $x_{0}=-2$ is zero of the polynomial

$$
p_{4}(-2)=(-2)^{4}-4(-2)^{3}-(-2)^{2}+16(-2)-12=16+32-4-32-12=0
$$

So $x_{0}=-2$ is zero of the polynomial and the polynomial has a factor of $x+2$.
2. We check, if the divisor of $x_{0}=-1$ is zero of the polynomial

$$
p_{4}(-1)=(-1)^{4}-4(-1)^{3}-(-1)^{2}+16(-1)-12=1+4-1-16-12=-32 \neq 0 .
$$

So $x_{0}=-1$ is not zero of the polynomial.
3. We check, if the divisor of $x_{0}=1$ is zero of the polynomial

$$
p_{4}(1)=(1)^{4}-4(1)^{3}-(1)^{2}+16(1)-12=1-4-1+16-12=0
$$

So $x_{0}=1$ is zero of this polynomial and the polynomial has a factor $x-1$.
4. We check, if the divisor of $x_{0}=2$ is zero of this polynomial

$$
p_{4}(2)=(2)^{4}-4(2)^{3}-(2)^{2}+16(2)-12=16-32-4+32-12=0
$$

So $x_{0}=2$ is zero of this polynomial and the polynomial has a factor $x-2$.
5 . We check, if the divisor of $x_{0}=3$ is zero of this polynomial

$$
p_{4}(3)=(3)^{4}-4(3)^{3}-(3)^{2}+16(3)-12=81-108-9+48-12=0
$$

So $x_{0}=3$ is zero of this polynomial and the polynomial contains a factor $x-3$.
Answer: The polynomial $p_{4}(x)$ has the following decomposition into linear factors

$$
p_{4}(x)=x^{4}-4 x^{3}-x^{2}+16 x-12=(x+2)(x-1)(x-2)(x-3) .
$$

Question 10.16 Find decomposition of the following polynomials

1. quadratic polynomial

$$
p_{2}(x)=2 x^{2}+6 x+4
$$

2. cubic polynomial

$$
p_{3}(x)=\left(x^{3}-8\right)+\left(x^{2}-4\right)
$$

3. polynomial of degree $n=4$.

$$
p_{4}(x)=x^{4}+6 x^{3}+12 x^{2}+11 x+6
$$

### 10.4.7 Polynomial inequalities

The topics Linear and Quadratic Functions describe ways to solve linear and square inequalities. Now we will deal with the solution of higher degree inequalities $n \geq 3$.
Consider the following inequality:

$$
p_{n}(x)=a_{n} x^{n}+a_{n-1} x^{n-1}+\cdots+a_{1} x+a_{0} \geq 0 \quad n \geq 1, \quad a_{n} \neq 0
$$

Solving the above inequality, we perform the following steps:

1. We factorize this polynomial

$$
p_{n}(x)=a_{n}\left(x-x_{1}\right)\left(x-x_{2}\right) \ldots .\left(x-x_{k}\right) q_{n-k}(x), \quad a_{n} \neq 0 .
$$

In the above distribution we allow $k$ real roots including multiple roots, $x_{1}, x_{2}, \ldots, x_{k}$. Note that .zeli $k=n$ is the polynomial $p_{n}(x)$ decomposes into lion factors and has all real roots $x_{1}, x_{2}, \ldots, x_{n}$.
Here, $q_{n-k}(x)$ is a non-linearly decomposable polynomial of degree $n-k$. This means that the polynomial $q_{n-k}(x)$ contains only square factors of the form $x^{2}+b x+c$ with the discriminant $\Delta=b^{2}-4 c<0$ negative.
2. We see that the inequality

$$
p_{n}(x)=a_{n}\left(x-x_{1}\right)\left(x-x_{2}\right) \ldots\left(x-x_{k}\right) q_{n-k}(x) \geq 0, \quad a_{n} \neq 0
$$

it is equivalent to inequality

$$
p_{n}(x)=\left(x-x_{1}\right)\left(x-x_{2}\right) \ldots\left(x-x_{k}\right) q_{n-k}(x) \geq 0, \quad \text { when } \quad a_{n}>0
$$

or equivalent to inequality

$$
p_{n}(x)=\left(x-x_{1}\right)\left(x-x_{2}\right) \ldots\left(x-x_{k}\right) q_{n-k}(x) \leq 0, \quad \text { when } \quad a_{n}<0 .
$$

Because we can always divide both sides of the inequality by $a_{n} \neq 0$ non-zero, keeping the direction of the inequality when $a_{n}>0$ is positive and changing the return of the inequality when $a_{n}<0$ is negative.
3. The solution can be read from the graph of the function

- The case of $a_{n}>0$ and all zeros of the polynomial $x_{1}, x_{2}, \ldots, x_{k}$ are different $x_{i} \neq x_{j}$ for $i \neq j$.
In the figure, an example of inequalities for a polynomial

$$
p_{5}(x)=2 x^{5}-x^{4}-10 x^{3}+5 x^{2}+8 x-4 \geq 0, \quad a_{5}=2>0
$$

We factorize this polynomial

$$
p_{5}(x)=(x+2)(x+1)\left(x-\frac{1}{2}\right)(x-1)(x-2) \geq 0
$$

We read zeros $x_{1}=-2, x_{2}=-1, x_{3}=\frac{1}{2}, x_{4}=1, x_{5}=2$


We read the solution from the grath, i.e. those intervals in which the polynomial is nonnegative:
Thus, this inequality holds true for $x \in[-2,-1] \cup\left[\frac{1}{2}, 1\right] \cup[2, \infty]$

- The case of $a_{n}<0$ and all zeros of the polynomial $x_{1}, x_{2}, \ldots, x_{k}$ are different $x_{i} \neq x_{j}$ for $i \neq j$.
In the figure, for example, the inequality for a polynomial

$$
p_{5}(x)=-2 x^{5}+x^{4}+10 x^{3}-5 x^{2}-8 x+4 \geq 0, \quad a_{5}=-2<0 .
$$

We factorize this polynomial

$$
p_{5}(x)=-2(x+2)(x+1)\left(x-\frac{1}{2}\right)(x-1)(x-2) \geq 0
$$

Dividing both sides of this inequality by -2 , we get the opposite-equivalent inequality

$$
p_{5}(x)=(x+2)(x+1)\left(x-\frac{1}{2}\right)(x-1)(x-2) \leq 0
$$

We read zeros $x_{1}=-2, x_{2}=-1, x_{3}=\frac{1}{2}, x_{4}=1, x_{5}=2$ and mark these zeros in the figure below


Inequality for polyniomial $p_{5}(x) \leq 0$.

We read the solution from the grath, i.e. those intervals in which the polynomial is non-positive:
So this inequality holds true for $x \in[-\infty,-2] \cup\left[-1, \frac{1}{2}\right] \cup[1,2]$.

- Case where the polynomial has multiple zeros. Then the graph of the polynomial does not cross the $x$ axis, if the multiplicity is even $2,4,6 \ldots$;
However, if the multiplicity is not even, then the plot of the polynomial cuts the $x$ axis.
We will explain the case of multiple zeros using the following example:
Solve the inequality:

$$
p_{3}(x)=x^{3}-2 x^{2}+3 x-1 \geq 0
$$

We factorize this polynomial

$$
p_{3}(x)=(x-1)(x+1)^{2} \geq 0
$$

Then we read the zeros of $x_{1}=-1$, and the double zero $x_{2}=1$. Let us mark these zeros in the figure

inequality for the polynomial $p_{3}(x) \geq 0$. double root at $x=1$.

We read the solution from the grath, i.e. those intervals in which the polynomial is nonnegative.
So this inequality holds true for $x \in[-1, \infty]$

## Chapter 11

## Simplified multiplication formulas and Newton's binomial

Let us start with binomials and cubes

### 11.1 Binomials and cubes

1. The following formulas hold: ${ }^{1}$

$$
\begin{aligned}
(a \pm b)^{1} & =a \pm b, & & \text { bilomial degree } n=1 \\
(a \pm b)^{2} & =a^{2} \pm 2 a b+b^{2}, & & \text { bilomial degree } n=2 \\
(a \pm b)^{3} & =a^{3}-3 a^{2} * b+3 * a * b^{2} \pm b^{3}, & & \text { cubic polynomial } n=3
\end{aligned}
$$

The formulas for the square of the sum or difference are obtained by multiplying the binomial $a \pm b$ by itself.
Namely, let's calculate

$$
\begin{aligned}
(a+b)^{2}=(a+b)(a+b) & =a(a+b)+b(a+b) \\
& =a^{2}+a b+b a+b^{2} \\
& =a^{2}+2 a b+b^{2} \\
(a-b)^{2}=(a-b)(a-b) & =a(a-b)-b(a-b) \\
& =a^{2}-a b-b a+b^{2} \\
& =a^{2}-2 a b+b^{2}
\end{aligned}
$$

Example 11.1 For a fixed natural number $n$ find the natural numbers $a$ and $b$ such that

$$
n+a^{2}=b^{2}
$$

## Solution

By shifting $a^{2}$ to the right side with the opposite sign, we get

$$
n=b^{2}-a^{2}
$$

[^10]because
$$
b^{2}-a^{2}=(b-a)(b+a)
$$
to
$$
n=(b-a)(b+a)
$$

Then we decompose the given number $n$ into the product

$$
n=1 * n
$$

By accepting

$$
b-a=1 \quad \text { and } \quad b+a=n
$$

we calculate the solution

$$
a=\frac{n-1}{2} \quad \text { and } \quad b=\frac{n+1}{2}
$$

We check that

$$
\left(\frac{n+1}{2}\right)^{2}-\left(\frac{n-1}{2}\right)^{2}=n
$$

If $n$ has a $p>1 \quad i \quad p<n$ then we decompose the number $n$ into the product

$$
n=p * q
$$

By accepting

$$
b-a=p \quad \text { and } \quad b+a=q
$$

we calculate the solution

$$
a=\frac{q-p}{2} \quad \text { and } \quad b=\frac{p+q}{2}
$$

We check that

$$
\left(\frac{p+q}{2}\right)^{2}-\left(\frac{q-p}{2}\right)^{2}=p * q=n
$$

If $n$ has a different factor $p_{1}>p i p_{1}<n$ then we decompose the number $n$ into the product

$$
n=p_{1} * q_{1}
$$

By accepting

$$
b-a=p_{1} \quad \text { and } \quad b+a=q_{1}
$$

we calculate the solution

$$
a=\frac{p_{1}-q_{1}}{2} \quad \text { and } \quad b=\frac{p_{1}+q_{1}}{2}
$$

We check that

$$
\left(\frac{p_{1}+q_{1}}{2}\right)^{2}-\left(\frac{p_{1}-q_{1}}{2}\right)^{2}=p_{1} * q_{1}=n
$$

In general, we factorise $n$ into $m$ prime factors

$$
n=p_{0} * p_{1} * p_{2} * \cdots * p_{m}, \quad p_{0}=1
$$

and then using the above decomposition into the product

$$
n=p_{k} * q_{k}
$$

we calculate the solution

$$
a=\frac{p_{k}-q_{k}}{2} \quad \text { and } \quad b=\frac{p_{k}+q_{k}}{2}
$$

We check that

$$
\left(\frac{p_{k}+q_{k}}{2}\right)^{2}-\left(\frac{p_{k}-q_{k}}{2}\right)^{2}=p_{k} * q_{k}=n
$$

for $k=0,1,2,3, \ldots, m$
This way we get all $m+1$ of natural solutions.
Example 11.2 Let $n=15$. Then we have a decomposition into a product

$$
15=1 * 15 \quad \text { or } \quad p=3 * 5
$$

So we have the first solution for $p=1$ and $q=15$

$$
b=\frac{15+1}{2}=8 \quad \text { and } \quad a=\frac{15-1}{2}=7
$$

We check that

$$
b^{2}-a^{2}=\left(\frac{15+1}{2}\right)^{2}-\left(\frac{15-1}{2}\right)^{2}=64-49=15
$$

Then for decomposition

$$
15=3 * 5
$$

we acceptp ${ }_{1}=3$ and $q_{1}=5$.
Then we get

$$
b+a=5 \quad \text { and } \quad b-a=3
$$

Hence we get the second solution

$$
b=\frac{5+3}{2}=4 \quad \text { and } \quad a=\frac{5-3}{2}=1
$$

We check that

$$
b^{2}-a^{2}=\left(\frac{5+3}{2}\right)^{2}-\left(\frac{5-3}{2}\right)^{2}=16-1=15
$$

2. 

$$
(a+b)^{3}=a^{3}+3 a^{2} b+3 a b^{2}+b^{3}, \quad(a-b)^{3}=a^{3}-3 a^{2} b+3 a b^{2}-b^{3} .
$$

Similarly, we check the cube of the sum or the difference.
Namely, let's calculate

$$
\begin{aligned}
(a+b)^{3}=(a+b)(a+b)^{2} & =a(a+b)^{2}+b(a+b)^{2} \\
& =a\left(a^{2}+2 a b+b^{2}\right)+b\left(a^{2}+2 a b+b^{2}\right) \\
& =\left(a^{3}+2 a^{2} b+a b^{2}\right)+\left(b a^{2}+2 a b^{2}+b^{3}\right) \\
& =a^{3}+3 a^{2} b+3 a b^{2}+b^{3}, \\
(a-b)^{3}=(a-b)(a-b)^{2} & =a(a-b)^{2}-b(a-b)^{2} \\
& =a\left(a^{2}-2 a b+b^{2}\right)-b\left(a^{2}-2 a b+b^{2}\right) \\
& =\left(a^{3}-2 a^{2} b+a b^{2}\right)-\left(b a^{2}-2 a b^{2}+b^{3}\right) \\
& =a^{3}-3 a^{2} b+3 a b^{2}-b^{3},
\end{aligned}
$$

## 3. Sum of squares

The sum of squares of two real numbers is equal to zero if and only if both numbers are equal zero,. If one of them is equal zero, then the sum is positive
Namely, we write

$$
\begin{aligned}
& a^{2}+b^{2}>0, \quad \text { when } \quad a \neq 0 \text { or } b \neq 0 \\
& a^{2}+b^{2}=0, \quad \text { when } \quad a=0 \text { and } b=0
\end{aligned}
$$

## 4. Difference of squares.

Difference of squares of two real numbers can be decomposed into linear factors as below

$$
a^{2}-b^{2}=(a-b)(a+b)
$$

We check the decomposition by multiplication

$$
(a-b)(a+b)=a(a+b)-b(a+b)=\left(a^{2}+a b\right)-\left(b a+b^{2}\right)=a^{2}-b^{2}
$$

5. Sum of cubes.

The sum of cubes of two real numbers is decomposed into the following product

$$
\left.a^{3}+b^{3}=(a+b)\right)\left(a^{2}-a b+b^{2}\right)
$$

We check this decomposition by multiplication

$$
\begin{aligned}
(a+b)\left(a^{2}-a b+b^{2}\right) & =a\left(a^{2}-a b+b^{2}\right)+b\left(a^{2}-a b+b^{2}\right) \\
& =\left(a^{3}-a^{2} b+a b^{2}\right)+\left(b a^{2}-a b^{2}+b^{3}\right)=a^{3}+b^{3}
\end{aligned}
$$

## 6. Difference of cubes.

The difference of cubes of two real numbers is decomposed into the product

$$
\left.a^{3}-b^{3}=(a-b)\right)\left(a^{2}+a b+b^{2}\right)
$$

We test this decomposition multiplication

$$
\begin{aligned}
(a-b)\left(a^{2}+a b+b^{2}\right) & =a\left(a^{2}+a b+b^{2}\right)-b\left(a^{2}+a b+b^{2}\right) \\
& =\left(a^{3}+a^{2} b+a b^{2}\right)-\left(b a^{2}+a b^{2}+b^{3}\right)=a^{3}-b^{3}
\end{aligned}
$$

### 11.1.1 Examples

Example 11.3 Perform arithmetic operations on algebraic expresions
(i) $(2 a+3)^{2}$,
(ii) $\left(\frac{x}{2}-4\right)^{2}$,
(ii) $(3 a+2)^{3}$,
(iv) $(2 x-3 y)^{3}$,

Solution. Using the formulas, we calculate

$$
\begin{aligned}
\text { ad. }(i) \quad(2 a+3)^{2} & =(2 a)^{2}+2(2 a) 3+3^{2}=4 a^{2}+12 a+9 . \\
\text { sd. }(i i) \quad\left(\frac{x}{2}-4\right)^{2} & =\left(\frac{x}{2}\right)^{2}-2\left(\frac{x}{2}\right)(-4)+(-4)^{2} \\
& =\frac{x^{2}}{4}-4 x+16 .
\end{aligned}
$$

$$
\begin{aligned}
\text { ad.(iii) } \quad(3 a+2)^{3} & =(3 a)^{3}+3(3 a)^{2} 2+3(3 a) 2^{2}+2^{3} \\
& =27 a^{3}+54 a^{2}+36 a+27 . \\
\text { ad.(iv) } \quad(2 x-3 y)^{3} & =(2 x)^{3}-3(2 x)^{2}(3 y)+3(2 x)(-3 y)^{2}-(3 y)^{3} \\
& =8 x^{3}-36 x^{2} y+54 x y^{2}-27 y^{3} .
\end{aligned}
$$

Question 11.1 Perform arithmetic operations on algebraic expresions
(i) $\frac{(5 a+2)^{2}}{(2 a-3)^{2}}$,
(ii) $\quad\left(\frac{x^{2}}{3}-1\right)^{2}$,
(ii) $(3 a+2)^{3}$,
(iv) $\left(\frac{3 x-2 y}{2 x+3 y}\right)^{3}$,

Question 11.2 Simplify the algebraic expression

$$
\begin{aligned}
& \text { (i) } \frac{\left(a^{2}+b^{2}\right)\left(a^{3}-b^{3}\right)\left(a^{3}+b^{3}\right)}{\left[(a+b)^{2}+(a-b)^{2}\right]\left(a^{2}+a b+b^{2}\right)\left(a^{2}-a b+b^{2}\right)} \\
& \text { (ii) } \frac{1+x+x^{2}+x^{3}}{1+x^{2}}
\end{aligned}
$$

### 11.2 Newton's binomial (1642-1727).

Newton's binomial takes the following form

$$
\begin{equation*}
(a+b)^{n}=\binom{n}{0} a^{n} b^{0}+\binom{n}{1} a^{n} b^{1}+\binom{n}{2} a^{n-2} b^{2}+\cdots+\binom{n}{n-1} a^{1} b^{n-1}+\binom{n}{n} a^{0} b^{n} \tag{11.1}
\end{equation*}
$$

or in $\Sigma$ (sigma) notation

$$
\begin{equation*}
(a+b)^{n}=\sum_{k=0}^{n}\binom{n}{k} a^{n-k} b^{k} \tag{11.2}
\end{equation*}
$$

where Newton's coefficients

$$
\binom{n}{k}=\frac{n!}{(n-k)!* k!}=\frac{(k+1)(k+2)(k+3) \cdots * n}{1 * 2 * 3 \cdots *(n-k+1)(n-k)}
$$

Let us write Newton's binomial for $n=1,2,3,4,5$

$$
\begin{array}{ll}
(a+b)^{1}=\sum_{k=0}^{1}\binom{1}{k} a^{1-k} b^{k}=a+b, & n=1 \\
(a+b)^{2}=\sum_{k=0}^{2}\binom{2}{k} a^{2-k} b^{k}=a^{2}+2 a b+b^{2}, & n=2 \\
(a+b)^{3}=\sum_{k=0}^{3}\binom{3}{k} a^{3-k} b^{k}=a^{3}+3 a^{2} b+3 a b^{2}+b^{3}, & n=3 \\
(a+b)^{4}=\sum_{k=0}^{4}\binom{4}{k} a^{4-k} b^{k}=a^{4}+4 a^{3} b+6 a^{2} b^{2}+4 a b^{3}+b^{4}, & n=4 \\
(a+b)^{5}=\sum_{k=0}^{5}\binom{5}{k} a^{5-k} b^{k}=a^{5}+5 a^{4} b+10 a^{3} b^{2}+10 a^{2} b^{3}+5 a b^{4}+b^{5}, & n=5
\end{array}
$$

Properties of Newton's coefficients $\binom{n}{k}$.

1. $\binom{n}{0}=\binom{n}{n}=1$.
2. Symmetry of Newton's coefficients

$$
\binom{n}{k}=\binom{n}{n-k}
$$

Indeed, we calculate that

$$
\binom{n}{n-k}=\frac{n!}{(n-k)!(n-(n-k))!}=\frac{n!}{k!(n-k)!}=\binom{n}{k}
$$

3. Sum of Newton's coefficients

$$
\binom{n}{k}+\binom{n}{k+1}=\binom{n+1}{k+1}
$$

let us observe that

$$
\begin{aligned}
\binom{n}{k} & =\frac{n!}{k!(n-k)!}=\frac{n!}{(k+1)!(n-k-1)!} * \frac{k+1}{n-k} \\
\binom{n}{k+1} & =\frac{n!}{(k+1)!(n-k-1)!}
\end{aligned}
$$

Summing up the above e quations by both sides, we get

$$
\begin{aligned}
\binom{n}{k}+\binom{n}{k+1} & =\frac{n!}{(k+1)!(n-k-1)!} * \frac{k+1}{n-k}+\frac{n!}{(k+1)!(n-k-1)!} \\
& =\frac{n!}{(k+1)!(n-k-1)!} *\left(\frac{k+1}{n-k}+1\right) \\
& =\frac{n!}{(k+1)!(n-k-1)!} * \frac{n+1}{n-k} \\
& =\frac{(n+1)!}{(k+1)!(n-k)!} \\
& =\binom{n+1}{k+1}
\end{aligned}
$$

4. Sum of Newton's binomial coefficients

$$
\begin{gathered}
\sum_{k=0}^{n}\binom{n}{k}=2^{n} \\
2^{n}=(1+1)^{n}=\sum_{k=0}^{n}\binom{n}{k} 1^{n-k} 1^{k}=\sum_{k=0}^{n}\binom{n}{k} .
\end{gathered}
$$

### 11.3 Pascal's triangle (1623-1662).

Pascal's triangle is formed by Newton's binomial coefficients.
$\begin{array}{lllllll}(a+b)^{0} & \ldots & \ldots & \ldots & \ldots & \ldots & \binom{1}{0}\end{array}$
$\begin{array}{llllll}(a+b)^{1} & \cdots & \cdots & \cdots & \cdots & \binom{1}{0}\end{array} \quad\binom{1}{1}$
$\begin{array}{lllll}(a+b)^{2} & \cdots & \cdots & \cdots & \binom{2}{0}\end{array} \quad\binom{2}{1} \quad\binom{2}{2}$
$(a+b)^{3} \quad \ldots \quad \ldots \quad\binom{3}{0} \quad\binom{3}{1} \quad\binom{3}{2} \quad\binom{3}{3}$
$\begin{array}{llll}(a+b)^{4} & \ldots & \cdots & \binom{4}{0}\end{array} \quad\binom{4}{1} \quad\binom{4}{2} \quad\binom{4}{3} \quad\binom{4}{4}$
$\begin{array}{llll}(a+b)^{5} & \cdots & \binom{5}{0} & \binom{5}{1}\end{array}\binom{5}{2}$
$(a+b)^{6} \quad \ldots \quad\binom{6}{0} \quad\binom{6}{1} \quad\binom{6}{2}$
$\begin{array}{llll}(a+b)^{7} & \binom{7}{0}\end{array}\binom{7}{1} \quad\binom{7}{2} \quad\binom{7}{3}$
$\binom{5}{5}$

$\binom{7}{3}$
$\binom{7}{4}$
$\binom{7}{5}$

Calculating the values of Newton's binomial coefficients from the formula

$$
\binom{n}{k}=\frac{n!}{k!(n-k)!}
$$

which are given in the table below

$$
\left.\begin{array}{cccccccccccl}
(a+b)^{0} & \ldots & \ldots & \ldots & \ldots & \cdots & 1 & & & & & \\
(a+b)^{1} & \ldots & \ldots & \ldots & \cdots & 1 & & 1 & & & \\
(a+b)^{2} & \ldots & \ldots & \ldots & 1 & & 2 & & 1 & & \\
(a+b)^{3} & \ldots & \ldots & 1 & & 3 & & 3 & & 1 & & \\
(a+b)^{4} & \ldots & \cdots & 1 & & 4 & & 6 & & 4 & & 1
\end{array}\right)
$$

The properties of Newton's coefficients can be easily read from the tables above. Namely, the property 1

$$
\binom{n}{0}=\binom{n}{n}=1 .
$$

is visible because the extreme values in each row are equal to 1 . The property 2 , symmetry, is also visible in the table

$$
\binom{n}{k}=\binom{n}{n-k} .
$$

To create a table of the values of Newton's coefficients in the nth row, use the property 3, which is the formula.

$$
\binom{n}{k}+\binom{n}{k+1}=\binom{n+1}{k+1}
$$

For example, from the values already calculated in the line $n-1$ let's calculate the values in the line $n$, as follows

$$
\begin{aligned}
& n=1, \quad k=0, \quad\binom{1}{0}+\binom{1}{1}=1+1=2=\binom{2}{1} \\
& n=2, \quad k=0, \quad\binom{2}{0}+\binom{2}{1}=1+2=3=\binom{3}{1} \\
& n=2, \quad k=1, \quad\binom{2}{1}+\binom{2}{2}=2+1=3=\binom{3}{2}
\end{aligned}
$$

## Chapter 12

## Linear functions

### 12.1 Straight line in a plane

Let us consider the plot of the linear function


The position of geometric figures and their shape, including the positions of lines on the Cartesian plane, are determined in the coordinates $x, y$.
Straight lines on the Cartesian plane are defined by linear equations that determine the dependence of the $y$ coordinate on the $x$ coordinate of points lying on the straight lines.
Let us consider the following four forms of equations:

- Equation of a line in the form of a linear function
- The equation of the line pass through two points
- General equation of the line.
- Parametric equation of a straight line


### 12.2 Linear functions.

Linar dependence

$$
\begin{equation*}
y(x)=a x+b, \tag{12.1}
\end{equation*}
$$

of coordinate $y$ from the coordinate $x$ is called a linear function with the direction coefficient $a$ and free term $b$ and the variable $x$.

The function $y(x)=a x+b$ is linear, because its plot is a straight line with the slope $a$ and the intercept $b$.

The equations of a straight line defined by the linear function

$$
y(x)=a x+b
$$

does not include lines parallel to the axis $y$.

## Example 12.1 .

(i) Draw a straight line on the plane in the coordinate system $x, y$, passing through the two points $(0,-1)$ and $(2,1)$
(ii) Calculate the coefficients of the linear function

$$
y(x)=a x+b
$$

when its grath passes through the points $(0,-1)$ and $(2,1)$

## Solution (i)



Plot of the linear function $y(x)=x-1$, in cartesian cordinates $x, y$

Solution (ii)
The graph of the function $y(x)=a x+b$ goes through the points $(0,-1),(2,1)$, if

$$
y(0)=-1, \quad y(2)=1
$$

Then the coordinates of these points satisfy the equations

$$
\begin{array}{ll}
y(0)=a * 0+b=-1, & b=-1, \\
y(2)=a * 2+b=1, & a * 2-1=1, \\
2 * a=2, & a=1
\end{array}
$$

From there we get the equation of the line

$$
y(x)=x-1
$$

in the form of a linear function with coefficients $a=1, b=-1$, on which the given points $(0,-1)$ and $(2,1)$.

Example 12.2 . (i) Check which of the points

$$
\begin{array}{ll}
P_{1}=(0.0), & P_{2}=(1,1), \\
P_{3}=(0,1), & P_{4}=(1,0)
\end{array}
$$

they lie on the straight lines $L_{1}$ or $L_{2}$ about the equations

$$
\begin{equation*}
L_{1}: \quad y_{1}(x)=x, \quad L_{2}: \quad y_{2}(x)=1-x \tag{12.2}
\end{equation*}
$$

(ii) Find the intersection point of the lines $L_{1}, L_{2}$. Plot these lines.

Solution $(i)$. The points $P_{1}=(0.0), P_{2}=(1,1)$ lie on the line $L_{1}$ because their coordinates satisfy the equation of the line $L_{1}: y=x$

$$
y(0)=0, \quad y(1)=1
$$

The points $P_{3}=(0,1), P_{2}=(1,0)$ lie on the line $L_{2}$ because their coordinates satisfy the equation of the line $L_{2}: y=1-x$,

$$
y(0)=1-0=1, \quad y(1)=1-1=0
$$

Solution (ii).
The intersection of $\left(x_{0}, y_{0}\right)$ lies on both lines, if

$$
y_{1}\left(x_{0}\right)=y_{0}, \quad \text { and } \quad y_{2}\left(x_{0}\right)=y_{0}
$$

Then we have the equations

$$
\begin{array}{lll}
y_{1}\left(x_{0}\right)=x_{0}=y_{0} & i & y_{2}\left(x_{0}\right)=1-x_{0}=y_{0} \\
x_{0}=1-x_{0} & i & 2 x_{0}=1 \\
x_{0}=\frac{1}{2} & i & y_{0}=\frac{1}{2}
\end{array}
$$

Answer: The lines $y_{1}(x)=x$ and $y_{2}(x)=1-x$ intersect at $\left(\frac{1}{2}, \frac{1}{2}\right)$


Point of intersection; of the perpendicular lines : $y_{1}(x)=x, \quad y_{2}(x)=1-x$.
Answer: The lines $y_{1}(x)=x$ and $y_{2}(x)=1-x$ intersect at $\left(\frac{1}{2}, \frac{1}{2}\right)$

### 12.3 Equations of parallel lines

Let us consider two lines $L_{1}$ and $L_{2}$ with equations ${ }^{1}$

$$
\begin{array}{ll}
L_{1}: & y=a_{1} x+b_{1} \\
L_{2}: & y=a_{2} x+b_{2} \tag{12.3}
\end{array}
$$

## Necessary and sufficient condition.

The lines $L_{1}$ and $L_{2}$ with the equations (12.3) are parallel if and only if the coefficients $a_{1}, a_{2}$ are equala $=a_{2}$

Example 12.3 Check if lines

$$
\begin{array}{ll}
L_{1}: & y=x+1, \\
L_{2}: & y=x-1 \tag{12.4}
\end{array}
$$

are parallel.
Plot the $L_{1}$ and $L_{2}$ lines.

## Solution.

Simple $L_{1}$ and $L_{2}$ with coefficients

$$
\begin{array}{ll}
a_{1}=1, & b_{1}=1 \\
a_{2}=1, & b_{2}=-1
\end{array}
$$

they are parallel because their coefficients $a_{1}, a_{2}$ satisfy the necessary and sufficient condition for parallelism of lines in the plane.

$$
a_{1}=a_{2}=1
$$



Example 12.4 Find the equation for the line $L$ parallel to the straight line

$$
L_{0}: \quad y=x+1
$$

[^11]passing through the point
$$
P=(3,1)
$$

Plot the line $L_{0}$ and the line $L$.

## Solution.

The straight line $L$ parallel to the line $L_{0}$ has the same slope as the line $L_{0}$, namely $a=1$. Then the line equation

$$
L: \quad y=x+b
$$

Since the line $L$ passes through the point $P=(3,1)$, then after substituting the coordinates of the point we will obtain the equation

$$
1=3+b
$$

from which we calculate the intercept

$$
b=1-3=-2
$$

Hence, we get the straight line equation

$$
L: \quad y=x-2
$$

Plot of parallel lines $L_{0}: y=x+1, \quad L: y=x-2$


Let us consider two lines $L_{1}$ and $L_{2}$ with equations

$$
\begin{array}{ll}
L_{1}: & y=a_{1} x+b_{1}, \\
L_{2}: & y=a_{2} x+b_{2} . \tag{12.5}
\end{array}
$$

## Necessary and sufficient condition.

The line $L_{1}$ is perpendicular to the line $L_{2}$, if and only if the coefficient $a_{2}$ of the line $L_{2}$ is equal to the negative reciprocal of the coefficient $a_{1}$ of the line $L_{1}$

$$
a_{2}=-\frac{1}{a_{1}}
$$

Then a line

$$
\begin{equation*}
y=-\frac{1}{a_{1}} x+b \tag{12.6}
\end{equation*}
$$

is a perpendicular to the line $L_{1}$ for any value of the intercept $b$.

Example 12.5 Check if the lines

$$
\begin{array}{ll}
L_{1}: & y=x-1, \\
L_{2}: & y=1-x \tag{12.7}
\end{array}
$$

are perpendicular.
Plot the $L_{1}$ and $L_{2}$ lines.

## Solution.

Simple $L_{1}$ and $L_{2}$ with coefficients

$$
\begin{array}{ll}
a_{1}=1, & b_{1}=1, \\
a_{2}=-1, & b_{2}=1
\end{array}
$$

are perpendicular because their coefficients $a_{1}, a_{2}$ satisfy the necessary and sufficient condition (12.6) of perpendiculars on the plane.

$$
a_{2}=-\frac{1}{a_{1}}=-\frac{(1)}{1}=-1
$$

Plot of lines $L_{1}, L_{2}$ with equations (??)


$$
L_{0}: \quad y=x+2
$$

passing through the point

$$
P=(2,-2)
$$

Plot the line $L_{0}$ and the line $L$.

## Solution.

The straight line $L$ is perpendicular to the line $L_{0}$ has a slope that is equal to the negative inverse of the factor $a=1$ of the line $L_{0}$.
Then the line equation

$$
L: \quad y=-\frac{1}{(1)} x+b=b-x
$$

Since the line $L$ passes through the point $P=(2,0)$ then the coordinates of this point satisfy the equation

$$
0=2+b
$$

from which we computed the intercept

$$
b=2
$$

Hence, we obtain the equation of the line

$$
L: \quad y=2-x
$$

### 12.5 Equation of a line passing through two points

${ }^{2}$ The equation for a straight line that passes through two given points does not include lines perpendicular to the $x$ axis.
The equation of a line passing through two different points with coordinates

$$
\left(x_{0}, y_{0}\right),\left(x_{1}, y_{1}\right), \quad \text { for } \quad x_{0} \neq x_{1}
$$

we write as the following dependence of the $y$ coordinate on the $x$ coordinate:

$$
\begin{equation*}
y=\frac{x-x_{1}}{x_{0}-x_{1}} y_{0}+\frac{x-x_{0}}{x_{1}-x_{0}} y_{1} \tag{12.8}
\end{equation*}
$$

Indeed, when $x=x_{0}$ then $y=y_{0}$ or when $x=x_{1}$ then $y=y_{1}$.
This means that the points $\left(x_{0}, y_{0}\right),\left(x_{1}, y_{1}\right)$ lie on a straight line.
Example 12.7 Write the equation for a line that passes through two points

$$
\left(x_{0}, y_{0}\right)=(-1,0) \quad \text { and } \quad\left(x_{1}, y_{1}\right)=(0,1)
$$

Find out which of the $(1,1),(1,2)$ points is on the straight.

## Solution

We write the equation of a line passing through the points

$$
\left(x_{0}, y_{0}\right)=(-1,0) \quad i \quad\left(x_{1}, y_{1}\right)=(0,1)
$$

Substituting to the formula (12.8) their coordinates we find the equation of the straight line

$$
\begin{aligned}
y & =\frac{x-x_{1}}{x_{0}-x_{1}} y_{0}+\frac{x-x_{0}}{x_{1}-x_{0}} y_{1} \\
& =\frac{x-0}{-1-0} * 0+\frac{x+1}{0+1} * 1 \\
& =x+1
\end{aligned}
$$

Answer: Equation of a straight line through the points (-1.0) and (0.1)

$$
y=x+1
$$

The point $(1,1)$ does not lie on the line $y=x+1$ because its coordinates do not satisfy the equation of this line because

$$
1 \neq 1+1
$$

The point $(1,2)$ lies on the line $y=x+1$ because its coordinates satisfy the equation of this line

$$
2=1+1
$$

[^12]

Note that the equations of the line defined by the linear function

$$
y(x)=a x+b
$$

or the line defined by two different points do not include the positions of the lines perpendicular to the $x$ axis. the position of the straight line on the plane is considered in the next section.

### 12.6 General equation of a straight line in a plane

General equation of a line on the plane

$$
\begin{equation*}
a x+b y+c=0, \quad a^{2}+b^{2}>0 \tag{12.9}
\end{equation*}
$$

where the factors $a, b$ do not disappear simultaneously for $a^{2}+b^{2}>0$.
Example 12.8 equation coefficients

$$
x+y-1=0
$$

$a=1, b=1, c=-1$

$$
a^{2}+b^{2}=1^{2}+1^{2}=2>0
$$

does not disappear at the same time The equation of this line can be written as a linear function

$$
y=1-x
$$



Let us consider three positions of the line $L$ about the equation

$$
a x+b y+c=0, \quad a^{2}+b^{2}>0
$$

1. The line $L$ is parallel to the $x$ axis if the coefficient $a=0$, and the coefficient $b \neq 0$. Then a straight line about the equation
it is parallel to the $x$ axis
2. The line $L$ is perpendicular to the $x$ axis if the coefficient $b=0$, and the coefficient $a \neq 0$. Then the line equation

$$
a x+c=0 \quad \text { or } \quad x=-\frac{c}{b}, \quad \text { for } \quad-\infty<y<\infty
$$

is perpendicular to the xaxis.

3. A straight line $L$ defined by the equation

$$
a x+b y+c=0, \quad \text { when } \quad a \neq 0, \text { and } b \neq 0
$$

intersects the $x$ axis at the point $\left(-\frac{c}{a}, 0\right)$ and the $y$ axis at the point $\left(0,-\frac{c}{b}\right)$
Example 12.9 Find the points of intersection of the line

$$
x+y-1=0
$$

with the $x$ axis and with the $y$ axis. Plot the line.

## Solution.

For the line $L$ with the coefficients $a=1, b=1, c=-1$ we calculate $x$-coordinate of the point of intersection of the line $x+y-1=0$ with the $x$ axis, when $y=0$

$$
x=-\frac{c}{a}=-\frac{(-1)}{1}=1
$$

the coordinate of the point of intersection of the line $x+y-1=0$ with the $y$ axis, when $x=0$

$$
y=-\frac{c}{b}=-\frac{(-1)}{1}=1
$$

The plot of the line with the equation $x+y-1=0$.


### 12.7 Parallel lines. General equation.

Let us consider two lines $L_{1}$ and $L_{2}$ with equations in general form

$$
\begin{array}{ll}
L_{1}: & a_{1} x+b_{1} y+c_{1}=0 \\
L_{2}: & a_{2} x+b_{2} y+c_{2}=0 \tag{12.10}
\end{array}
$$

Straight lines $L_{1}$ and $L_{2}$ with the equations (??) are parallel if the coefficients $a_{1}, b_{1}$ are proportional to the coefficients $i b_{1}, b_{2}$, that is

$$
\begin{equation*}
a_{1}=k * a_{2}, \quad b_{1}=k * b_{2} \tag{12.11}
\end{equation*}
$$

for some $k \neq 0$, which we call the aspect ratio.
Example 12.10 Let us check, if the lines

$$
\begin{array}{ll}
L_{1}: & x-y+1=0 \\
L_{2}: & x-y-1=0 \tag{12.12}
\end{array}
$$

are parallel.
Provide the plot of the straight line $L_{1} i L_{2}$.

## Solution.

Lines $L_{1}$ and $L_{2}$ are parallel because their coefficients

$$
\begin{array}{ll}
a_{1}=1, & b_{1}=-1, \\
a_{2}=1, & b_{2}=-1,
\end{array}
$$

satisfy the proportion condition (12.11)

$$
1=1 * 1, \quad-1=-1 * 1
$$

for the aspect ratio $k=1$
Plots of equations (12.12)


Note that the line about the equation

$$
a x+b y+c=0
$$

- intersects exis $y$, in he point $\left(0,-\frac{c}{b}\right)$, when $x=0$, then the line is paralell to the exis $x$

$$
b y+c=0, \quad \text { and } \quad y=-\frac{c}{b}, \quad \text { for } \quad b \neq 0, \quad-\infty<y<\infty
$$

- intersects the exis $x$, in the point $\left(-\frac{c}{a}, 0\right)$, when $y=0$, then the line is psaralell to the exis y

$$
a x+c=0, \quad \text { and } \quad x=-\frac{c}{a} \quad \text { for } \quad a \neq 0, \quad-\infty<x<\infty
$$

- two straights lines are given by the equations

$$
\begin{array}{ll}
L_{1}: & a_{1} x+b_{1} y+c_{1}=0 \\
L_{2}: & a_{2} x+b_{2} y+c_{2}=0 \tag{12.13}
\end{array}
$$

intersect at the point $\left(x_{0}, y_{0}\right)$, if this point satisfies the equations of these lines

$$
\begin{array}{ll}
L_{1}: & a_{1} x_{0}+b_{1} y_{0}+c_{1}=0 \\
L_{2}: & a_{2} x_{0}+b_{2} y_{0}+c_{2}=0 \tag{12.14}
\end{array}
$$

Example 12.11 Find the position of two lines with equations on the $(x, y)$ plane

$$
\begin{aligned}
& x-y=0 \\
& x+y-1=0
\end{aligned}
$$

Find their intersection points with the axes $x$ and $y$ and the point of intersection of these lines.

Solution. The line with the equation $x y=0$ intersects at ś $x$ io ś $y$, if $y=0$, or $x=0$, then $x=y=0$. So this line goes through the origin of the coordinate system, through the point (0.0).
The line with the equation $x+y-1=0$ intersects by 's $x$ when $y=0$. Then we have the equation

$$
x-1=0, \quad \text { and } x=1 .
$$

The line with the equation $x+y-1=0$ intersects the $y$ axis when $x=0$. Then we have the equation

$$
y-1=, \quad \text { and } y=1
$$

Thus, this line intersects 's $x$ at the point $(1,0)$ and intersects the $y$ axis at the point $(0,1)$. Two straight lines intersect at the point $\left(x_{0}, y_{0}\right)$, when the coordinates of this point satisfy both equations, i.e.

$$
\begin{aligned}
& x_{0}-y_{0}=0, \quad y_{0}=x_{0} \\
& x_{0}+y_{0}-1=0
\end{aligned}
$$

Substituting $y_{0}=x_{0}$ into the second equation we find

$$
x_{0}+y_{0}-1=0, \quad 2 x=1, \quad x=\frac{1}{2}, \quad y=\frac{1}{2} .
$$

We see that the lines intersect at $\left(\frac{1}{2}, \frac{1}{2}\right)$


Plot of the lines: $y=x, \quad y=1-x$

### 12.8 Perpendicular lines. General equation

Let us consider two lines $L_{1}$ and $L_{2}$ with equations in general form

$$
\begin{array}{ll}
L_{1}: & a_{1} x+b_{1} y+c_{1}=0 \\
L_{2}: & a_{2} x+b_{2} y+c_{2}=0 \tag{12.15}
\end{array}
$$

The lines $L_{1}$ and $L_{2}$ with equations (??) are perpendicular if and only if the coefficient factors $a, b_{1}$ and $a_{2}, b_{2}$ satisfy the equation

$$
\begin{equation*}
a_{1} * b_{1}+a_{2} * b_{2}=0 \tag{12.16}
\end{equation*}
$$

Example 12.12 Check if the lines

$$
\begin{array}{ll}
L_{1}: & 2 x-y-2=0 \\
L_{2}: & x+2 y+2=0 \tag{12.17}
\end{array}
$$

are perpendicular.
Provide plot of lines $L_{1} i L_{2}$.

## Solution.

The lines $L_{1}$ and $L_{2}$ are perpendicular below their coefficients

$$
\begin{array}{ll}
a_{1}=2, & b_{1}=-1 \\
a_{2}=1, & b_{2}=2
\end{array}
$$

satisfy the proportion condition(12.16)

$$
2 * 1+(-1) * 2=0
$$

Graphs of lines $L_{1}$ and $L_{2}$ defined by equations(12.27)


### 12.9 Parametric equation of straight line

Parametric equation of a line $L$ going through two points

$$
P=\left(x_{1}, y_{1}\right) \quad \text { and } \quad Q=\left(x_{2}, y_{2}\right)
$$

we write in the form

$$
\begin{equation*}
L(t)=P+(Q-P) t, \quad-\infty<t<+\infty \tag{12.18}
\end{equation*}
$$

or in the for

$$
\begin{equation*}
L(t)=Q * t+(1-t) P, \quad-\infty<t<+\infty \tag{12.19}
\end{equation*}
$$

Note that the points $P$ and $Q$ lie on the line $L(t)$, because for the parameter $t=0$ we have the point

$$
L(0)=P
$$

and for the parameter $t=1$ we have the point

$$
L(1)=Q
$$

If the $t$ parameter changes from 0 to 1 , then the point $L(t)$ changes along the segment beginning at $P$ and ending at $Q$. However, if the $t$ parameter changes from $-\infty$ to $+\infty$, then $L(t)$ runs through the entire straight line $L$.
Then the line $L$ is parallel to the vector

$$
\vec{v}=Q-P
$$

about coordinates

$$
\vec{v}=\left(x_{2}-x_{1}, y_{2}-y_{1}\right)
$$

The parametric equation of the line $L(t)$ is also written in coordinates

$$
\begin{align*}
& x(t)=x_{1}+t * x_{2} \\
& y(t)=y_{1}+t * y_{2} \tag{12.20}
\end{align*}
$$

for the paramiter $t \in(-\infty,+\infty)$.
Example 12.13.
(i) Find the parametric equation of $L(t)$ passing through two points $P=(0,-1)$ i $Q=(2,1)$
(ii) Give the graph of the line $L(t)$.

Solution (i)
Substituting the given points into the parametric line equation (12.19)

$$
P=(0,-1) \quad i \quad Q=(2,1)
$$

we get the equation

$$
\begin{equation*}
L(t)=L(t)=(2,1) t+(1-t)(0,-1) \tag{12.21}
\end{equation*}
$$

We write the equation (12.20) in coordinates

$$
\begin{align*}
& x(t)=2 t  \tag{12.22}\\
& y(t)=2 t-1
\end{align*}
$$

for the paramiter $t \in(-\infty,+\infty)$.


### 12.10 Questions

Question 12.1.
(i) Draw a straight line on the plane in the coordinate system $x, y$, that passes through the two points $(-1,-2)$ and $(2,1)$
(ii) Calculate the coefficients of the linear function

$$
y(x)=a x+b,
$$

passing through the points $(-1,-2)$ and $(2,1)$
Question 12.2 Give the position on the plane $(x, y)$ of two straight lines $L_{1}$ and $L_{2}$ with equations

$$
L_{1}: y=2 x-1, \quad L_{2}: y=1-2 x
$$

Find the intersection of $L_{1}$ and $L_{2}$.
Question 12.3 Write the equation of a line going through two points $\left(x_{0}, y_{0}\right)=(-1,-1)$ $i\left(x_{1}, y_{1}\right)=(1.1)$. Check which of the points $(0,1),(2,2)$ lies on a straight.

Question 12.4
(i) Check which of the points

$$
\begin{array}{ll}
P_{1}=(0.0), & P_{2}=(1,1), \\
P_{3}=(0,2), & P_{4}=(2,0)
\end{array}
$$

they lie on straight lines $L_{1}$ or $L_{2}$ with equations

$$
\begin{equation*}
L_{1}: \quad y_{1}(x)=2 x, \quad L_{2}: \quad y_{2}(x)=2-x \tag{12.23}
\end{equation*}
$$

(ii) Find the intersection point of the lines $L_{1}, L_{2}$. Give the graph of these lines.

Question 12.5 Check if the lines with the equation

$$
\begin{array}{lll}
L_{1}: & y=3 x+1, & L_{3}: \\
L_{2}: & y=3 x+3  \tag{12.24}\\
& & \\
& L_{4}: & 3 x-3
\end{array}
$$

are paralell.
Provide the graphs of the straight line $L_{1} i L_{2}$.
Question 12.6 Find the equation for the line $L$ parallel to the straight line

$$
L_{0}: \quad y=1-x
$$

passing through the point

$$
P=(-1,1)
$$

Give the graph of the line $L_{0}$ and the line $L$
Question 12.7 Check if the straight lines

$$
\begin{array}{ll}
L_{1}: & y=0.5 x-1 \\
L_{2}: & y=1-2 x \tag{12.25}
\end{array}
$$

are perpendicular.
Provide the plots of the straight line $L_{1}$ and $L_{2}$.

Question 12.8 Find the equation for the line $L$ perpendicular to the straight line

$$
L_{0}: \quad y=2-x
$$

passing through the point

$$
P=(-1,-1)
$$

Plot of the line $L_{0}$ and the line $L$
Question 12.9 Write the equation for a line that passes through two points

$$
\left(x_{0}, y_{0}\right)=(-1,2) \quad i \quad\left(x_{1}, y_{1}\right)=(0,1)
$$

Find out which of the points $(1,0),(2,-1)$ is on the straight.
Question 12.10 Find the coefficients $a, b, c$ of the equation of the line $L$ in the general form

$$
L: \quad a x+b y+c=0
$$

going through the points

$$
P=(-2,2), \quad Q=(1,0)
$$

Question 12.11 Plot and find the points of intersection of the line

$$
2 x+y-4=0
$$

with the $x$ axis and with the $y$ axis
Question 12.12 Check if the straight lines

$$
\begin{array}{ll}
L_{1}: & 2 x-y+1=0 \\
L_{2}: & 4 x-2 y-1=0 \tag{12.26}
\end{array}
$$

are paralell.
Provide the plots of the straight line $L_{1} i L_{2}$.
Question 12.13 Find the position of two lines with equations on the $(x, y)$ plane

$$
\begin{aligned}
& 2 x-y=0 \\
& x+2 y-1=0
\end{aligned}
$$

Find their intersection points with the axes $x$ and $y$ and the point of intersection of these lines.

Question 12.14 Check if the straight lines

$$
\begin{array}{ll}
L_{1}: & 3 x-y-1=0 \\
L_{2}: & x+3 y+1=0 \tag{12.27}
\end{array}
$$

are perpendicular.
Provide line graphs $L_{1} i L_{2}$.

## Question 12.15

(i) Find the parametric equation of $L(t)$ passing through two points $P=(1,-1)$ i $Q=$ $(2,-1)$
(ii) Plot the line $L(t)$.

Question 12.16 Find the parametric equation of the line with the equation

$$
y=x
$$

given in the coordinates $x, y$.
Question 12.17 Find the intersection of lines with parametric equations

$$
\begin{aligned}
& L_{1}(t): x(t)=t, \quad y(t)=t, \quad-\infty<t<\infty \\
& L_{2}(t): x(t)=t, \quad y(t)=-t, \quad-\infty<t<\infty
\end{aligned}
$$

in the plane in the coordinates $x, y$.

## Chapter 13

## Rational functions



### 13.1 Rational functions

A natural extension of the concept of polynomials are rational functions. Namely, fhe following quotient of polynomials of degrees $n$ and $m$ is a rational function.

$$
\begin{equation*}
w(x)=\frac{p_{n}(x)}{q_{m}(x)}=\frac{a_{n} x^{n}+a_{n-1} x^{n-1}+\cdots+a_{1} x+a_{0}}{b_{m} x^{m}+b_{m-1} x^{n-1}+\cdots+b_{1} x+b_{0}}, \quad q_{m}(x) \neq 0 \tag{13.1}
\end{equation*}
$$

Let us note that if the denominator $q_{m}(x)=$ constant $\neq 0$ is non-zero, the rational function is a polynomial of degree $n$.

Thus, the domain of a rational functions is the set of these real numbers

$$
x \in R=(-\infty, \infty)
$$

for which the denominator $q_{m}(x) \neq 0$ is different from zero.

$$
\text { Domain } w(x): \quad D=\left\{x \in(-\infty, \infty): \text { such that } q_{m}(x) \neq 0\right\}
$$

### 13.2 Examples of rational functions

We will consider some examples of standard rational functions below.

### 13.2.1 Hyperbola

The simplest rational function is the hyperbola

$$
y=\frac{1}{x}, \quad x \neq 0
$$

The domain of the hyperbola is the set

$$
D=\{x \in(-\infty, \infty): \quad x \neq 0 .\}
$$

The hyperbola $y=\frac{1}{x}$ attainds all its real values other than zero.
The values of the hyperbola $y=\frac{1}{x}$ tends to zero when $x \rightarrow \infty$.
The hyperbola $\frac{1}{x}$ has two asymptots: $x$ exis when $y=0$ and $y$ exes when $x=0$
Plot of the hyperbola in canonical form


Let us note that this hyperbola has two asymptotes: a horizontal $x$ axis, and a vertical yaxis.
Indeed, when $x$ tends toward positive infinity or negative infinity, we write

$$
x \rightarrow \pm \infty
$$

it is the values of the hyperbola tend to zero

$$
w(x)=\frac{1}{x} \rightarrow 0, \quad \text { when } \quad x \rightarrow \pm \infty
$$

Example 13.1 Let us consider a rational function

$$
y=\frac{x-1}{x+1}, \quad y=w(x) \quad x \neq-1
$$

For this rational function, which we denote hereinafter, $y=w(x)$, we will find

1. the domain,
2. the set of values,
3. fractional form of the function $y=w(x)$
4. asymptotes of the function $y=w(x)$,
5. graph of the function $y=w(x)$.

The domain of this rational function is the set of real numbers for which the denominator

$$
x+1 \neq 0
$$

is not zero.
Thus, the domain of this rational function is the set

$$
D=\{x: x \in(-\infty, \infty) \text { for which, } x \neq-1 .\}
$$

Let us write the rational function $w(x)$ in the form of simple fractions. Namely, by adding and deducting the number 2 in the numerator, we check that

$$
\begin{aligned}
y=w(x) & =\frac{x-1}{x+1} \\
& =\frac{x-1+2-2}{x+1} \\
& =\frac{(x+1)-2}{x+1} \\
& =1-\frac{2}{x+1}, \quad x \neq-1 .
\end{aligned}
$$

The set of values of the function

$$
y=w(x)=1-\frac{2}{x+1} \neq 1, \quad x \neq-1
$$

is the set of real numbers different from 1. So that we write

$$
R(D)=\{w \in(-\infty, \infty), \text { different from } w \neq 1\}
$$

The rational function $w(x)$ reaches all real values other than 1.
The asymptotes of the function $y=w(x)$ :
The horizontal asymptote is a straight line parallel to the $x$ axis

$$
y=w(x)=1 \quad \text { for } \quad \text { all real } \quad x \neq-1
$$

If $x$ goes to positive or negative infinity then the values of the functionin( $x$ ) go to 1 .

$$
\text { If } x \rightarrow \pm \infty, \text { to } y=w(x)=1-\frac{2}{x+1} \rightarrow 1
$$

The vertical asymptote is a straight line parallel to the yaxis through the singularity point $x=-1$.

If $x$ goes to -1 to the left or right of $x=-1$, then the values of the functionin( $x$ ) go to plus or minus infinity.

$$
x \rightarrow-1, \quad \text { to } y=w(x)=w(x)=1-\frac{2}{x+1} \rightarrow \pm \infty
$$

The graph of this rational function is hyperbola


Note that this hyperbola has two asymptotes: a horizontal $y=1$ for each real value of $x \in(-\infty, \infty)$ and a vertical one passing through the point $x=-1$, this is the point at which the function is undefined.

Example 13.2 Let us cosider the rational function

$$
y=w(x)=\frac{1}{16 x^{2}+1}, \quad y=w(x), \quad-\infty<x<\infty
$$

For the function wymiernej $y=w(x)$ we will find

1. the domain,
2. the set of values,
3. fractional form of the function $y=w(x)$
4. asymptotes of the function $y=w(x)$,
5. graph of the function $y=w(x)$.

The domain of this rational function is the set of real numbers.

$$
\text { The doimain of the functionis : } w(x): \text { is } D=(-\infty<x<, \infty)
$$

The set of values of this function is the range $[1, \infty)$ of real numbers greater than or equal to 1 . Indeed, we note that the values of this function satisfy the inequality

$$
\frac{1}{16 x^{2}+1} \geq 1, \quad \text { for } \quad-\infty<x<\infty
$$

The graph of this rational function is the curve

has one horizontal asymptote x exis $x$, when $y=0$.
Example 13.3 Let us consider rational function

$$
y=\frac{x^{2}-1}{x^{2}+1}, \quad y=w(x), \quad-\infty<x<\infty .
$$

For the function $w(x)$ we will findznajdziemy

1. the domain,
2. the set of values,
3. fractional form of the function $y=w(x)$
4. asymptotes of the function $y=w(x)$,
5. graph of the function $y=w(x)$.

The domain of this rational function is the set of all real numbers because the denominator $x^{2}+1>1$ is positive for every real

$$
x \in(-\infty, \infty)
$$

The set of function values is the $[-1,1)$ range of real numbers. Namely, we ubserve thatcheck

$$
\begin{equation*}
-1 \leq \frac{x^{2}-1}{x^{2}+1}<1 \tag{13.2}
\end{equation*}
$$

Indeed, the function $i n(x)$ can be written as a difference

$$
\frac{x^{2}-1}{x^{2}+1}=1-\frac{2}{x^{2}+1}
$$

Positive value of the expression

$$
0<\frac{2}{x^{2}+1} \leq 2
$$

is less than 2 , not 2 for $x=0$.
in addition

$$
0<\frac{2}{x^{2}+1} \rightarrow 0, \text { when } x \rightarrow \pm \infty
$$

goes to zero if $x$ rightarrow $\pm \infty$.

Hence, we get the inequality (13.2) by the following evaluation

$$
\begin{aligned}
& \frac{x^{2}-1}{x^{2}+1} \\
& =1-\frac{2}{x^{2}+1} \\
& \\
& \text { and } \\
& \begin{aligned}
\frac{x^{2}-1}{x^{2}+1} & =1-\frac{2}{x^{2}+1} \\
& \geq-1, \text { when } x \rightarrow \pm \infty \\
& x=0
\end{aligned}
\end{aligned}
$$

Plot of this rational function, below

$$
y=w(x)=\frac{x^{2}-1}{x^{2}+1}
$$



### 13.2.2 Decomposition of rational functions into simple fractions

A simple fraction is one of the following rational functions

$$
\frac{A}{x-a}, \quad \frac{A}{(x-a)^{k}}, \quad \frac{A x+B}{x^{2}+p x+q}, \quad \frac{A x+B}{\left(x^{2}+p x+q\right)^{k}}, \quad \Delta=p^{2}-4 q<0 .
$$

for certain natural number $k$, coefficients $A, B$ and $p, q$, when discriminand $\Delta=p^{2}-4 q<0$.
Example 13.4 Decompose a rational function into simple fractions

$$
w(x)=\frac{2 x-1}{x^{2}-1}
$$

For the function $w(x)$ we find

1. the set of values,
2. fractional form of the function $y=w(x)$
3. asymptotes of the function $y=w(x)$,
4. plot of the function $y=w(x)$.

Solution. The domain of this rational function is the set of real numbers for which the denominator is not zero.

$$
\begin{aligned}
D & =\left\{x \in(-\infty, \infty): x^{2}-1=(x-1)(x+1) \neq 0\right\} \\
& =\{x \in(-\infty, \infty):(x \neq 1) \cap(x \neq-1)\} .
\end{aligned}
$$

We look for the distribution of a rational function into simple fractions using the co-method 'o l indefinite factors. Namely, we find $A$ and $B$ such that the following equality holds

$$
w(x)=\frac{2 x-1}{x^{2}-1}=\frac{2 x-1}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}
$$

for every $x \in D$ from the function domain $w(x)$, that is for every $x \neq-1$ and $x \neq 1$. Thus, the coefficients $A$ and $B$ are derived from the identity

$$
\frac{2 x-1}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}
$$

which is true for every $x \neq-1$ and $x \neq 1$.
We will write this identity with a common denominator

$$
\frac{2 x-1}{(x-1)(x+1)}=\frac{A(x+1)+B(x-1)}{(x-1)(x+1)}=\frac{(A+B) x+(A-B)}{(x-1)(x+1)}
$$

Comparing the coefficients for $x$ and the intercepts in the numerator, we get the equations for the unknowns $A$ and $B$

$$
A+B=2, \quad A-B=-1
$$

We calculate

$$
A=B-1, \quad(B-1)+B=2, \quad 2 B=3 .
$$

Hence, we find

$$
B=\frac{3}{2}, \quad A=B-1=\frac{3}{2}-1=\frac{1}{2} .
$$

Answer: The decomposition of the rational function $\operatorname{in}(x)$ into simple forms

$$
w(x)=\frac{2 x+1}{x^{2}-1}=\frac{3}{2(x-1)}+\frac{1}{2(x+1)}
$$

### 13.3 Questions

Question 13.1 For the rational function

$$
w(x)=\frac{2}{x+2}, \quad x \neq-2 .
$$

find the following items

1. the domain of the function in $w(x)$,
2. a collection of function values $w(x)$,
3. as a fraction of a simple function $w(x)$,
4. asymptotes of the function $w(x)$,
5. Sketch a plot of $y=w(x)$.

Question 13.2 For the rational function
(i) $\quad w(x)=\frac{2 x-1}{x-2}$,
(ii) $\quad w(x)=\frac{x-2}{x^{2}-4}$,
give the followig items

1. the domain of the function in $(x)$,
2. collection of functionvaluesin $(x)$,
3. asymptotes of the functionin $(x)$, item Make fractions the function $w(x)$,
4. Sketch a plot of the $y=w(x)$ function.

Question 13.3 Decompose rational functions into simple fractions

$$
w(x)=\frac{x^{2}-9}{(x-3)\left(x^{2}+4\right)} e
$$

and give the following items

1. the domain of the function in $(x)$,
2. collection of functionvaluesin $(x)$,
3. asymptotes of the functionin $(x)$, item Make fractions the function $w(x)$,
4. Sketch a plot of the function $y=w(x)$.

## Chapter 14

## Exponential function

Let us consider the exponential function given by the following formula

$$
y=f(x)=a^{x}, \quad a>0, \quad a \neq 1 .
$$

The real number $a>0, a \neq 1$ positive and different from 1 is called the basis of the exponential function.
The domain of the exponential function is the set of all real numbers

$$
D=\{x \in R: \quad-\infty<x<\infty\} . \quad R=(-\infty<x<\infty)
$$

${ }^{1}$ Every exponential function attains all values from the set of positive real numbers.

$$
R_{+}=\{y \in R, \quad 0<y<\infty\}
$$

Let us note the domain and set of values of the exponential function

$$
f(x)=2^{x}, \quad \text { domain is : } R, \text { set of values is : } R_{+}, 0<2^{x}<\infty .
$$

These sets of real numbers are marked on the graph, below.


Let us observe, on the plot, that the exponential function has one asymptote which is $x$ exis. The function

$$
y=f(x)=a^{x}
$$

is ascending if its base $a>1$, and it is decreasing if its base $0<a<1$.

[^13]Now let us consider plot of the exponential function $y=f(x)=\left(\frac{1}{2}\right)^{x}$ with the base $0<a=\frac{1}{2}<1$.


We can see from the above grath that, the exponential function

$$
y=\left(\frac{1}{2}\right)^{x}
$$

is decreasing.
The following realtions between vaues of an exponential function hold.

1. The value of the exponential function at zero when $x=0$ is equal to one.

$$
y=f(0)=1, \quad \text { because } \quad a^{0}=1,
$$

for every base $a>0$.
2. The value of the exponential function for $x=1$ is equal the base $a$.

$$
y=f(1)=a, \quad \text { because } \quad a^{1}=a,
$$

3. Exponential function $y=f(x)$ from the sum of arguments is equal to the product of the values

$$
f(x+t)=f(x) * f(t)
$$

Indeed, we have

$$
f(x+t)=a^{x+t}=a^{x} * a^{t}=f(x) * f(t)
$$

4. The exponential function of the arguments difference is equal to the quotient of the values

$$
f(x-t)=\frac{f(x)}{f(t)}
$$

In fact, we have

$$
f(x-t)=a^{x-t}=a^{x} * a^{-t}=\frac{a^{x}}{a^{t}}=\frac{f(x)}{f(t)}
$$

5. The exponential function of the product of arguments is equal to the power

$$
f(x * t)=(f(x))^{t}
$$

Let us check

$$
f(x * t)=a^{x * t}=\left(a^{x}\right)^{t}=(f(x))^{t}
$$

6. The exponential function of $\frac{m}{n}$ is equal to the nth root of the mth power

$$
f\left(\frac{m}{n}\right)=\sqrt[n]{f(m)}
$$

Namelyy, we have

$$
f\left(\frac{m}{n}\right)=a^{\frac{m}{n}}=\sqrt[n]{a^{m}}=\sqrt[n]{f(m)}
$$

Example 14.1 Calculate vaule of the arithmetic expression

$$
3^{8} * 3^{-5}
$$

Solution:
In this example, we apply the property 2 to the exponential function

$$
f(x)=a^{x}
$$

when the base $a=3$ and the arguments $x=8$ and $x=-5$. So using the property 2 , we calculate

$$
f(3) * f(-5)=3^{8} * 3^{-5}=3^{8-5}=3^{3}=27
$$

Example 14.2 Calculate value of the arithmetic expression

$$
3^{\frac{5}{2}} * 12^{\frac{1}{2}}
$$

## Solution

Using the properties of the exponential function, we find

$$
\begin{aligned}
3^{\frac{5}{2}} * 12^{\frac{1}{2}} & =3^{\frac{5}{2}} *(3 * 4)^{\frac{1}{2}} \\
& =3^{\frac{5}{2}} * 3^{\frac{1}{2}} * 4^{\frac{1}{2}} \\
& =3^{\frac{5}{2}+\frac{1}{2}} * \sqrt{4}=3^{2} * 2=18
\end{aligned}
$$

Question 14.1 Calculate value of the arithmetic expression

$$
\text { (i) } \quad\left(3 \frac{1}{3}\right)^{-2}, \quad \text { (ii) } \quad 2^{\frac{8}{3}} * 2^{-\frac{5}{3}} * 16^{\frac{1}{2}}
$$

Question 14.2 Consider the exponetial function

$$
f(x)=2^{x}, \quad-\infty<x<\infty .
$$

Sketch a plot of the function

$$
y=f(x-1)+1, \quad-\infty<x<\infty
$$

in the coordinate system $x, y$
Calculate value of the fuinction $f(x-1)+1$ for $x=3$.

### 14.0.1 Exponential equations and inequalities

We solve exponential equations and inequalities using the following relations

- $f(x)=a^{x}>0$ is positive along the whole number line $-\infty<x<\infty$.
- set of values of the exponential function consits of all positive numbers, $R_{+}=(0, \infty)$.
- exponential function $f(0)=0$ for each base $a>0, a \neq 1$
- exponential function $f(x)=a^{x}$ increments along the entire number line $-\infty<x<\infty$, if the base $a>1$.
- exponential function $f(x)=a^{x}$ is decreasing along the whole number line $-\infty<x<\infty$ if base $0<a<1$.


### 14.0.2 Examples

Example 14.3 Solve the equation

$$
2^{2 x}-3 * 2^{x}+2=0
$$

Solution. The domain of this equation is the entire set of $R$ real numbers. Now, let's write this equation as

$$
\left(2^{x}\right)^{2}-3 * 2^{x}+2=0
$$

Using the substitution $t=2^{x}$ we get a quadratic equation

$$
t^{2}-3 t+2=0, \quad \Delta=(-3)^{2}-4 * 2=1
$$

we find the roots of this equation

$$
t_{1}=\frac{3-\sqrt{1}}{2}=1, \quad \frac{3+\sqrt{1}}{2}=2
$$

Returning to the variable $x$, we calculate the solution:

$$
\begin{aligned}
& \text { if } 2^{x}=1, \text { to } \quad x=0 \\
& \text { if } 2^{x}=2, \text { to } \quad x=1
\end{aligned}
$$

Example 14.4 Solve the equation

$$
3^{\frac{2 x-1}{3 x-1}}=9
$$

Solution. The domain of this equation is set of real numbers different from $\frac{1}{3}$. Now, let's write this equation as follows

$$
3^{\frac{2 x-1}{3 x-1}}=3^{2}
$$

Hence, we get the equation

$$
\frac{2 x-1}{3 x-1}=2
$$

and we find the solution

$$
2 x-1=2(3 x-1), \quad 2 x-1=6 x-2, \quad 4 x=1, \quad x=\frac{1}{4}
$$

Question 14.3 Solve the equation

$$
3^{x}+27 * 3^{-x}-12=0
$$

Question 14.4 Solve the equation

$$
5^{\frac{3 x-1}{2 x-3}}=25
$$

## Chapter 15

## Arithmetic root $\sqrt[n]{a}$

The operation to calculate the root of degree $n$ from a real numbera is the inverse operation to exponentiation. But the reverse operation not always is possible in real numbers.

Let's start by calculating the arithmetic square root.
Definicja 15.1 The square root of a non-negative number $a \geq 0$ is the non-negative number $b \geq 0$, which satisfies the equality

$$
b^{2}=a
$$

The square root of $a \geq 0$ is denoted by the symbol

$$
b=\sqrt{a} .
$$

Example 15.1 The square root of $a=4$ is $b=2$ because the number is positive 2 and satisfies

$$
2^{2}=4
$$

We write

$$
\sqrt{4}=2 .
$$

Also, negative number - 2 satisfies equality

$$
(-2)^{2}=4,
$$

However, by definition, -2 is not the square root of 4 .

In general, real roots of even degrees

$$
n=2 k, \quad k=1,2,3, \ldots:
$$

do not exist of negative numbers. In particular, the square root of negative numbers does not exist in the set of real numbers.

### 15.1 Square root function

Similarly, we define the square root function.
Definicja 15.2 The square root function

$$
y=\sqrt{x}
$$

is the square root of a non-negative real $x \geq 0$.
So the square root function is well defined for the argument $x \in[0, \infty)$ and its the values of $y \in[0, \infty)$ belong to the interval $[0, \infty)$.


Example 15.2 Simplify the expression by decomposing a number under a square into prime factors
(i) $\sqrt{200}$,
(ii) $\sqrt{144}$

## Solution.

(i)

$$
\sqrt{200}=\sqrt{2 * 100}=\sqrt{2 * 10^{2}}=10 \sqrt{2}
$$

(ii)

$$
\sqrt{432}=\sqrt{3 * 144}=\sqrt{3 * 12^{2}}=12 \sqrt{3}
$$

Example 15.3 Calculate the value of the expression

$$
\begin{aligned}
\frac{(10-\sqrt{10})(10+\sqrt{10})}{\sqrt{10}} & =\frac{100-10}{\sqrt{10}} \\
& =\frac{90}{\sqrt{10}} \left\lvert\, * \frac{\sqrt{10}}{\sqrt{10}}\right. \\
& =\frac{90 \sqrt{10}}{10} \\
& =9 \sqrt{10}
\end{aligned}
$$

Example 15.4 Simplify the expression by factorization the numbers under the root

$$
\sqrt{432}-\sqrt{48}
$$

## Solution.

In order to decompose numbers 432 and 48 on prime factors we write

| 432 | 2 | 48 | 2 |
| ---: | ---: | ---: | ---: |
| 216 | 2 | 24 | 2 |
| 108 | 2 | 12 | 2 |
| 54 | 2 | 6 | 2 |
| 27 | 3 | 3 | 3 |
| 9 | 3 | 1 |  |
| 3 | 3 |  |  |
| 1 |  |  |  |

Hence, we get the decomposition of numbers into prime factors

$$
432=2^{4} * 3^{3}, \quad 48=2^{4} * 3
$$

Now, let us do simplification of the expression

$$
\begin{aligned}
\sqrt{432}-\sqrt{48} & =\sqrt{2^{4} * 3^{3}}-\sqrt{2^{4} * 3} \\
& =3 \sqrt{16 * 3}-\sqrt{16 * 3}=2 \sqrt{3}
\end{aligned}
$$

Example 15.5 Simplify the expression

$$
\begin{aligned}
\frac{\sqrt{90}-\sqrt{40}}{\sqrt{10}} & =\frac{\sqrt{9 * 10}-\sqrt{4 * 10}}{\sqrt{10}} \\
& =\frac{\sqrt{3^{2} * 10}-\sqrt{2^{2} * 10}}{\sqrt{10}} \\
& =\frac{3 \sqrt{10}-2 \sqrt{10}}{\sqrt{10}} \\
& =\frac{\sqrt{10}}{\sqrt{10}}=1
\end{aligned}
$$

### 15.2 Algorithm digit by digit to calculate the square root

Let's start the description of the algorithm with examples.
Example 15.6 Calculate the approximate value of the root $\sqrt{2}$ with the accuracy of 4 decimal places.

The scheme of the algorithm for computing the square root of $a=2.0>0$ is similar to the scheme for dividing integers.

1. In the first step, digits of $a=2.0$ are completed with zeros and divided into groups of two to the left of the decimal point and to the right of the decimal point, as shown below

$$
\sqrt{02.00000000}
$$

2. We find the largest number $p$ such that $p^{2}$ is less than the first two digits of $a$. In this example

$$
p^{2} \leq a=2
$$

Sure, for $a=2$ number $p=1$ because $p^{2}=1^{2}<2$.
Thus, the number $p=1$ is written above the dash, as shown below

$$
\begin{array}{ccc}
1 . & \text { digits } \\
\sqrt{02.00000000} \quad \mid & 1
\end{array}
$$

Subtract $p * 1=1 * 1=1$ from 02 as in written division

| 1. |  | cyfry $\sqrt{a}$ |
| :--- | :--- | :--- |
| $\sqrt{02,00000000}$ |  |  |
| 01 | $r_{1}=100$ | 1 |
| --- |  |  |

3.The next digit of the number $\sqrt{2}$ is found by adding to the number $2 * p=2 * 1$ the one digit $x$ for which the product

$$
\begin{equation*}
y=(20 p+x) * x \leq r_{1}=100 \tag{15.1}
\end{equation*}
$$

In this way we increase the digits of $p$ by one digit $x$ which we compute, in this example, by substituting $p=4$ into the equation (15.1)

$$
y=(20 * 4+4) * 4=96 .
$$

The number 4 is added to the number 1. above the decimal point, then we perform the subtraction operation as in written division

4. We find the next digit of $p=1.4$ in a similar way.

Namely, the number $p=14$ is multiplied by 2 and added to the product of digits $x$ for which the value of the expression

$$
(20 p+x) * x=(20 * 14+1) * 1=281 \leq 400
$$

is the largest and less than 400 . We can easily check that $x=1$.
Digit $x=1$ is added to the number $p=1.4$ above the line. Next we perform the subtraction operation as in the written division

| 1.41 |  | cyfry $\sqrt{a}$ |
| :--- | :--- | :--- |
| $\sqrt{02,00000000}$ |  |  |
| 01 | $r_{1}=100$ | $x=1$ |
| --- | $r_{2}=20 * 4+4=96$ | $x=4$ |
| 100 |  |  |
| 96 | $r_{3}=(20 * 14+1) * 1=281$ | $x=1$ |
| -- |  |  |
| 400 |  |  |

The number 4 is added to the number 1. above the decimal point, then we perform the subtraction operations as in written division

5. We find the next digit of $p=1.41$ in a similar way.

Namely, we multiply the number $p=141$ by 2 and add to the product the digit $x$ for which the value of the expression

$$
(20 p+x) * x=(20 * 141+4) * 4=11256 \leq 11900
$$

is the largest and less than 11900 . We can easily check that $x=4$.
The number $x=4$ is added to the number $p=1.41$ above the line. Next we perform the subtraction operation as in the written division

| 1.414 |  | cyfry $\sqrt{a}$ |
| :--- | :--- | :--- |
| $\sqrt{02,00000000}$ |  |  |
| 01 | $r_{1}=100$ | $x=1$ |
| --- | $r_{2}=20 * 4+4=96$ | $x=4$ |
| 100 |  |  |
| 96 | $r_{3}=(20 * 14+1) * 1=281$ | $x=1$ |
| -- | $r_{3}=(20 * 14+1) * 1=281$ | $x=4$ |
| --- |  |  |

6. We find the next digit of $p=1.414$ in a similar way.

Namely, we multiply the number $p=1414$ by 2 and add to the product the digit $x$ for which the value of the expression

$$
(20 p+x) * x=(20 * 1414+2) * 2=56,564 \leq 60,400
$$

is the largest, but less than 60,400 . We can easily check that $x=2$.
The digit $x=2$ is added to the number $p=1,414$ above the line. Next we perform the subtraction operations as in the written division

| 1.4142 |  | $c y f r y \sqrt{a}$ |
| :---: | :---: | :---: |
| $\sqrt{02,00000000}$ |  |  |
| 01 |  |  |
| - - | $r_{1}=100$ | $x=1$ |
| 100 |  |  |
| 96 | $r_{2}=20 * 4+4=96$ | $x=4$ |
|  |  |  |
| $400$ |  |  |
| 281 | $r_{3}=(20 * 14+1) * 1=281$ | $x=1$ |
| - - - |  |  |
| 11900 |  |  |
| 11296 | $r_{3}=(20 * 141+1) * 1=281$ | $x=4$ |
| - - - |  |  |
| 60400 |  |  |
| 56564 | $r_{3}=(20 * 1414+2) * 2=56564$ | $x=2$ |
| - - - |  |  |
| 3836 |  |  |

Finishing the calculations with an accuracy of 4 digits after the decimal point, we get the approximate value of the root $\sqrt{2} \approx 1.4142$.
Sure, we can continue with this $\sqrt{2}$ process to get more precision than 4 .

### 15.2.1 Equations with root $\sqrt{x}$

Let us explaine solution of equations with $\sqrt{x}$ on examples.

Example 15.7 Solve the equation:

$$
x=\sqrt{x}, \quad x \geq 0
$$

Solution. Naturally, we are looking for a solution in the domain of this equation, that is, in the range $[0, \infty)$ non-negative numbers. Squaring this equation of both the sides, we get an equation not equivalent

$$
\begin{equation*}
x^{2}=x, \quad-\infty<x<\infty, \tag{15.2}
\end{equation*}
$$

which makes numerical sense for all real numbers including negative numbers.
We find a solution easily

$$
x-x^{2}=0, \quad x(x-1)=0, \quad x=0
$$

or

$$
\begin{equation*}
x-1=0, \quad x=1 . \tag{15.3}
\end{equation*}
$$

which makes numerical sense for all real numbers including negative numbers.
Let's check that both the roots of $x=0$ or $x=1$ belong to the $[0, \infty)$ domain. Thus this equation has two solutions to $x=0, x=1$.

Example 15.8 Solve the equation

$$
\begin{equation*}
\sqrt{2 x}=\sqrt{x-1} \tag{15.4}
\end{equation*}
$$

## Solution

Note that the equation (15.4) is defined for the root expression $2 x \geq 0$ when $x \geq 0$ and for the right-hand expression $x-1 \geq 0$, when $x \geq 1$.
The domain of this equation is the ray $[1, \infty)$.
Squaring the equation (15.4) by sides, we get an equation not equivalent

$$
2 x=x-1,
$$

which has the solution

$$
x=-1
$$

does not belong to the domain of equation (15.4), write $x=-1 \notin[1, \infty)$.
Answer: Equation (15.4) has no real solutions.
Example 15.9 Solve the equation:

$$
\begin{equation*}
\sqrt{x+1}-\sqrt{x-1}=1, \quad x \geq 1 \tag{15.5}
\end{equation*}
$$

Solution. Naturally, we are looking for the solution in the domain of this equation, that is in the range $(1, \infty)$, when $x+1 \geq 0$ and $x-1 \geq 0$.

Squaring this equation with the sides, we get an equation not equivalent

$$
\begin{equation*}
(x+1)-2 \sqrt{(x+1)(x-1)}+(x-1)=1 \tag{15.6}
\end{equation*}
$$

lub
which makes numerical sense for all real numbers $x \leq-1$ or $x \geq 1$ including negative numbers less than -1 . Thus, equation (15.5) has a domain different from that of equation (15.6).
Let's write the equation (15.6) as

$$
\sqrt{x^{2}-1}=\frac{1}{2}-x, \quad x \geq 1
$$

Further, by squaring the last equation once again, we get an equation not equivalent as well

$$
x^{2}-1=\left(\frac{1}{2}-x\right)^{2}
$$

lub

$$
x^{2}-1=\frac{1}{4}-x+x^{2},
$$

lub,

$$
x-\frac{5}{4}=0
$$

which makes numerical sense for all real numbers.
The solution to the last equation is $x=\frac{5}{4}>1$ which belongs to the inherited equation.
We check that $x=\frac{5}{4}$ is the solution of the equation (15.5)

$$
\sqrt{\frac{5}{4}+1}-\sqrt{\frac{5}{4}-1}=1, \quad \sqrt{\frac{9}{4}}-\sqrt{\frac{1}{4}}=\frac{3}{2}-\frac{1}{2}=1
$$

### 15.3 Cubic root $\sqrt[3]{a}$

Unlike the roots of even degrees, there are real negaive roots of odd degrees

$$
n=2 k+1, k=1,2,3, \ldots
$$

from negative numbers.

Definicja 15.3 The cubic root, whenn $=3$, of positive or negative number $a$ is a number $b=\sqrt[3]{a}$ which satisfies equality

$$
b^{3}=a
$$

For example, for $a=8$ or $a=-8$ the cubic root

$$
\begin{array}{lll}
b=\sqrt[3]{8}=2, & \text { because } & b^{3}=2^{3}=8 \\
b=\sqrt[3]{-8}=-2 & \text { because } & b^{3}=(-2)^{3}=-8
\end{array}
$$

The cubic roots of some numbers are given in the table below

| a | -125 | -64 | -27 | -8 | -1 | 0 | 1 | 8 | 27 | 64 | 125 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $y=\sqrt[3]{a}$ | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 |

### 15.4 Cubic root function $y=\sqrt[3]{x}$

Like the square root function, we define the cubic root function.
Definicja 15.4 The value of $y$ of the function cubic root

$$
y=\sqrt[3]{x}
$$

is equal to the cubic root of $x$, where $x \in(-\infty, \infty)$.
Thus, the cubic root function is well defined for the $x \in[-\infty, \infty)$ argument and the $y \in(-\infty, \infty)$ values belonging to the set of real numbers $(-\infty, \infty)$.

Note that the root function is increasing, that is, it has larger values for larger arguments, we write If arguments $x_{1}, x_{2}$ satisfy the inequality

$$
x_{1}<x_{2}
$$

then the corresponding values of $y_{1}, y_{2}$ satisfy the inequality

$$
y_{1}<y_{2} .
$$



### 15.5 Examples

Example 15.10 Calculate the value of the expression

$$
\frac{\sqrt[3]{81}}{\sqrt[3]{64}}
$$

Solution.
Let us observe that $81=3^{3}$ and $64=2^{6}$.
We calculate

$$
\frac{\sqrt[3]{81}}{\sqrt[3]{64}}=\frac{\sqrt[3]{3^{3}}}{\sqrt[3]{2^{6}}}=\frac{3}{4}
$$

Example 15.11 Calculate the value of the expression

$$
\frac{\sqrt[3]{81}-\sqrt[3]{64}}{\sqrt[3]{3}-4}
$$

## Solution.

Clearly

$$
81=3^{4}, \quad 64=2^{6}
$$

Hence we find the value of the expression

$$
\frac{\sqrt[3]{81}-\sqrt[3]{64}}{3 \sqrt[3]{3}-4}=\frac{\sqrt[3]{3^{3}}-\sqrt[3]{2^{6}}}{3 \sqrt[3]{3}-4}=\frac{3 \sqrt[3]{3}-4}{3 \sqrt[3]{3}-4}=1
$$

Example 15.12 Simplify the expression by decomposing a number under a square into prime factors

$$
\text { (i) } \sqrt[3]{192}, \quad \text { (ii) } \sqrt[3]{648}
$$

## Solutionn.

(i)
(ii)

$$
\sqrt[3]{192}=\sqrt[3]{3 * 64}=\sqrt[3]{3 * 2^{6}}=\sqrt[3]{3 * 4^{3}}=4 \sqrt[3]{3}
$$

$$
\sqrt[3]{648}=\sqrt[3]{8 * 81}=\sqrt[3]{2^{3} * 3^{4}}=2 \sqrt[3]{3^{3} * 3}=2 * 3 \sqrt[3]{3}=6 \sqrt[3]{3}
$$

Example 15.13 Calculate value of the expression

$$
\frac{(100-\sqrt[3]{1000})(100+\sqrt[3]{1000})}{\sqrt[3]{1000}}
$$

## Solution.

Let us observe that $\sqrt[3]{1000}=10$ and applying formula for difference of squares

$$
\begin{aligned}
\frac{(100-\sqrt[3]{1000})(100+\sqrt[3]{1000})}{\sqrt[3]{1000}} & =\frac{\left(100-\sqrt[3]{10^{3}}\right)\left(100+\sqrt[3]{10^{3}}\right)}{10} \\
& =\frac{(100-10)(100+10)}{10} \\
& =\frac{100^{2}-10^{2}}{10}=\frac{10000-100}{10}=990
\end{aligned}
$$

### 15.6 Arithmetic root of degree $n$

Let us start with definition of arithmetic $n$-the root of a real number $a$.
Definicja 15.5 The arithmetic root of degree $n$ - th of a non-negative number $a \geq 0$ is the nonnegative number $b \geq 0$, which satisfies the equality

$$
b^{n}=a, \quad n=2,3,4, \ldots ;
$$

The arithmetic root of $a \geq 0$ is denoted by the symbol

$$
b=\sqrt[n]{a}
$$

We give the arithmetic roots of some non-negative numbers below.
Example 15.14

$$
\text { for } n=2, \quad a=256, \quad \sqrt{256}=16, \quad b=16, \quad 16^{2}=256,
$$

$$
\text { for } n=3, \quad a=512, \quad \sqrt[3]{512}=8, \quad b=8, \quad 8^{3}=512,
$$

$$
\text { for } n=4, \quad a=256, \quad \sqrt[4]{256}=4, \quad b=4, \quad 4^{4}=256,
$$

$$
\text { for } n=5, \quad a=1024, \quad \sqrt[5]{1024}=4, \quad b=4, \quad 4^{5}=1024,
$$

### 15.7 Operations on arithmetic roots

The formula for arithmetic operations on roots is given in the table below

| $\sqrt[n]{a^{n}}=a$ | $a \geq 0$ | $a^{\frac{n}{n}}=a$ |
| :---: | ---: | ---: |
| $\sqrt[n]{a * b}=\sqrt[n]{a} * \sqrt[n]{b}$ | $a \geq 0$ | $b \geq 0$ |
| $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ | $a \geq 0$ | $b>0$ |
| $\sqrt[n]{a^{m}}=(\sqrt[n]{a})^{m}$ | $a \geq 0$ | $\sqrt[n]{a^{m}}=a^{\frac{\pi n}{n}}$ |

For example

| $\sqrt[n]{2^{n}}=2$ | $a=2 \geq 0$ | $2^{\frac{n}{n}}=2^{1}=2$ |
| :---: | ---: | ---: |
| $\sqrt[2]{4 * 9}=\sqrt[2]{4} * \sqrt[2]{9}=2 * 3=6$ | $a=4 \geq 0$ | $b=9 \geq 0$ |
| $\sqrt[3]{\frac{125}{64}}=\frac{\sqrt[3]{125}}{\sqrt[3]{64}}=\frac{5}{4}$ | $a=125 \geq 0$ | $b=64>0$ |
| $\sqrt[4]{3^{8}}=(\sqrt[4]{3})^{8}$ | $a=3 \geq 0$ | $\sqrt[4]{3^{8}}=3^{\frac{8}{4}}=3^{2}=9$ |

Example 15.15 Let us calculate value of the following expression

$$
\sqrt[3]{\sqrt[2]{4096}}=\sqrt[3]{\sqrt[2]{2^{12}}}=\sqrt[3]{2^{\frac{12}{2}}}=\sqrt[3]{2^{6}}=\sqrt[3]{\left(2^{2}\right)^{3}}=2^{2}=4
$$

### 15.8 Questions

Question 15.1 Calculate the value of the expression by decomposing the number under the square into prime factors

$$
\begin{array}{lll}
\text { (i) } & \sqrt{300}, & \text { (ii) } \\
\sqrt{169}
\end{array}
$$

Question 15.2 Calculate value of the expression

$$
\frac{(20-\sqrt{10})(20+\sqrt{10})}{\sqrt{3}}
$$

Question 15.3 Calculate the value of the expression by factoring the numbers under the root

$$
\sqrt{3072}
$$

Question 15.4 Simplify the expression

$$
\frac{\sqrt{160}-\sqrt{90}}{\sqrt{10}}
$$

Question 15.5 Calculate value of the expressiona

$$
\frac{\sqrt[3]{729}}{\sqrt[3]{512}}
$$

Question 15.6 Calculate the value of the expression by decomposing the number under the square into prime factors
(i) $\sqrt[3]{384}$,
(ii) $\sqrt[3]{1296}$

Question 15.7 Calculate value of the espression

$$
\frac{(20-\sqrt[3]{1000})(20+\sqrt[3]{1000})}{\sqrt[3]{1000}}
$$

Question 15.8 Calculate value of the espression

$$
\frac{\sqrt[3]{\sqrt{3^{6}}}}{\sqrt[3]{\sqrt{2^{6}}}}
$$

Question 15.9 Solve the equation

$$
\sqrt{x+1}=x
$$

Question 15.10 Solve the equation

$$
\sqrt{2 x-1}=1
$$

Question 15.11 Solve the equation

$$
\sqrt{x+2}-\sqrt{x-2}=2
$$

## Chapter 16

## Logarithmic function

The logarithmic function is inverse one to the exponential function. Thus, if the exponential function defines dependence of the variable $y$ on the variable $x$ by the formula

$$
y=a^{x}, \quad a>0, \quad a \neq 1
$$

then the inverse function determines the dependence of the $x$ variable on the $y$ variable by the formula

$$
x=\log _{a} y, \quad y>0 .
$$

The constant $a>0, \quad a \neq 1$ or $0<a<1$ is called the base of the logarithm.
Therefore the domain of a logarimic function is the set of values of the exponential function

$$
D=R_{+}=\{y: 0<y<\infty\}
$$

and the set of values of the logarithmic function is the domain of the exponent function

$$
R_{+}=\{x: 0<x<\infty\}
$$

For example, the decimal logarithm, when $a=10$ we write

$$
x=\log _{10} y, \quad \text { for } \quad y>0
$$

We swap the role of $x$ and $y$. Namely, we denote the independent variable with the letter $x$, while the dependent variable denoted by the letter $y$.

We write the decimal logarithm without the base 10 as below

$$
y=\log x, \quad x>0
$$

Logarithmic function monotonically increases for base greater than one $a>1$, and is decrement if base $0<a<1$.
Let us see the graths of the logarithmic function, when the base $a=2$ or $a=\frac{1}{2}$

$$
y=\log _{2} x, \quad 0<x<\infty
$$

or

$$
y=\log _{\frac{1}{2}} x \quad 0<x<\infty
$$




### 16.1 Natural logarithmic functions

The natural logaritmic function

$$
x=\ln n_{e} y \quad \text { or } \quad x=\ln y, \quad 0<y<\infty
$$

with the base

$$
a=e \approx e=2.71828182845904523536028747135266249775724709369995 \ldots ;
$$

is inverse to the expotential function

$$
y=e^{x}, \quad \text { or } \quad y=\operatorname{Exp}[x], \quad-\infty<x<\infty .
$$

### 16.1.1 Properties of logarithmic function

1. The value of the logarithmic function

$$
y=g(x)=\log _{a} x
$$

for $x=1$ is equal to zero.

$$
g(1)=\log _{a} 1=0, \quad \text { because } a^{0}=1, \quad a>0, \quad a \neq 1 .
$$

2. The value of the logarithmic function

$$
y=g(a)=\log _{a} x
$$

for $x=a$ is equal to one.

$$
g(a)=\log _{a}=1, \quad \text { poniewaz } a^{1}=a, a>0, \quad a \neq 1
$$

3. The logarithmic function of the product of arguments is equal to the sum of the values

$$
\log _{a} x * t=\log _{a} x+\log _{a} t, \quad x>0, \quad t>0, \quad a>0, \quad a \neq 1
$$

or we write

$$
g(x)=\log _{a} x, \quad g(x * t)=g(x)+g(t), \quad x>0, \quad t>0 .
$$

Indeed we check

$$
\begin{array}{ll}
y_{1}=\log _{a} x, & \text { to } \quad x=a^{y_{1}}, a>0, a \neq 1 \\
y_{2}=\log _{a} t, & \text { to } \quad t=a^{y_{2}}, a>0, a \neq 1
\end{array}
$$

Hence we find

$$
\begin{array}{cl}
x * t & =\quad a^{y_{1}} * a^{y_{2}}=a^{y_{1}+y_{2}}, \quad a>0, \quad a \neq 1 \\
\log _{a} x * t & =\log _{a} a^{y_{1}+y_{2}}=y_{1}+y_{2}=\log _{a} x+\log _{a} t
\end{array}
$$

4. The logarithmic function of the quotient of the arguments is equal to the difference of the values

$$
\log _{a} \frac{x}{t}=\log _{a} x-\log _{a} t, x>0, \quad t>0, \quad a>0, \quad a \neq 1 .
$$

or we write

$$
g(x)=\log _{a} x, g\left(\frac{x}{t}\right)=g(x)-g(t), x>0, t>0
$$

Indeed we check

$$
\begin{array}{lll}
y_{1}=\log _{a} x, & \text { to } & x=a^{y_{1}}, a>0, a \neq 1 \\
y_{2}=\log _{a} t, & \text { to } & t=a^{y_{2}}, a>0, a \neq 1
\end{array}
$$

Hence we find

$$
\begin{array}{ll}
\frac{x}{t} & =\frac{a^{y_{1}}}{a^{y_{2}}}=a^{y_{1}-y_{2}}, \quad a>0, \quad a \neq 1 \\
\log _{a} \frac{x}{t} & =\log _{a} a^{y_{1}-y_{2}}=y_{1}-y_{2}=\log _{a} x-\log _{a} t
\end{array}
$$

5. Logarithmic function from $x^{k}, k=0,1,2,, 3, \ldots$, is equal to the product of the power exponent $k$ times the logarithm of the base of the power $x$

$$
\log _{a} x^{k}=k * \log _{a} x, \quad x>0, \quad k=0,1,2,3, \ldots
$$

This relationship is a direct result of the logarithm of the product. Namely

$$
\log _{a} x^{k}=\underbrace{\log _{a} x * x * \cdots * x}_{k}=\underbrace{\log _{a} x+\log _{a} x+\cdots+\log _{a} x}_{k}=k * \log _{a} x
$$

6. The logarithmic function of $x^{\frac{m}{n}}$ is equal to the logarithm

$$
\log x^{\frac{m}{n}}=m * \log \sqrt[n]{x}
$$

Namely, we check using the relation of the logarithmic and exponential functions

$$
\log _{a} x^{\frac{m}{n}}=\underbrace{\log _{a} \sqrt[n]{x}+\log _{a} \sqrt[n]{x}+\cdots+\log _{a} \sqrt[n]{x}}_{m}=m * \log _{a} \sqrt[n]{x}
$$

7. Assuming $a>0, a \neq 1, c>0, c \neq 1, b>0$, we can change the $a$ basis of the $\log _{a} b$ arithm to $c$ according to the formula

$$
\log _{a} b=\frac{\log _{c} b}{\log _{c} a}
$$

In order to chech this formula we intrduce notation

$$
p=\log _{a} b, \quad q=\log _{c} b, \quad r=\log _{c} a
$$

From the definition we find

$$
b=a^{p}, \quad b=c^{q}, \quad a=c^{r}
$$

Hence follows equality

$$
\begin{array}{ll}
b=\left(c^{r}\right)^{p}, & b=c^{p * r} \\
\log _{c} b=p * r \log _{c} c, & \log _{c} c=1 \\
\log _{c} b=p * r, & \log _{c} b=\log _{a} b * \log _{c} a \\
\log _{a} b=\frac{\log _{c} b}{\log _{c} a}, &
\end{array}
$$

8. In the case of $c=b$ replacing the base with the logarithmic number $b$ leads to the reciprocal of the logarithm

$$
\log _{a} b=\frac{1}{\log _{b} a}
$$

In fact, with relaion 7 , for $c=b$ we have

$$
\log _{a} b=\frac{\log _{b} b}{\log _{b} a}=\frac{1}{\log _{b} a}, \quad \text { bo } \quad \log _{b} b=1
$$

Example 16.1 Calculate expresion

$$
\text { (i) } \quad \log _{2} 64, \quad \text { (ii) } \quad \log _{5} 125
$$

Directly from definition we find

$$
\begin{aligned}
& \text { (i) } \quad \log _{2} 64=\log _{2} 2^{6}=6, \quad \text { bo } \quad 2^{6}=64 \\
& \text { (ii) } \quad \log _{5} 125=\log _{5} 5^{5}=5 \quad \text { bo } \quad 5^{5}=125
\end{aligned}
$$

Example 16.2 Calculate the value of logarithmic expressions
(i) $\frac{\log _{3} 625}{\log _{3} 5}$,
(ii) $\frac{\log _{8} 5}{\log _{2} 5}$,
(iii) $\log _{2}\left(\log _{2} \sqrt{5}\right)-\log _{2}\left(\log _{2} 5\right)$,

Using the relationship between the values of the logarithms, we calculate
(i) $\frac{\log _{3} 625}{\log _{3} 5}=\frac{\log _{3} 5^{4}}{\log _{3} 5}=\frac{4 \log _{3} 5}{\log _{3} 5}=4$
(ii) $\frac{\log _{8} 5}{\log _{2} 5}=\frac{\log _{2} 5}{\log _{2} 8 \log _{2} 5}=\frac{1}{\log _{2} 2^{3}}=\frac{1}{3}$
(iii) $\log _{2}\left(\log _{2} \sqrt{5}\right)-\log _{2}\left(\log _{2} 5\right)=\log _{2} \frac{\log _{2} \sqrt{5}}{\log _{2} 5}=\frac{1}{2} * \frac{\log _{2} \sqrt{5}}{\log _{2} \sqrt{5}}=\log _{2} \frac{1}{2}=-1$

Example 16.3 Calculate the value of logarithmic expressions
(i) $\log _{2}\left(\log _{4} 16\right)$,
(ii) $\log _{3}\left(\log _{5} 125\right)$.

Using the relationship between the values of the logarithms, we calculate
(i) $\log _{2}\left(\log _{4} 16\right)=\log _{2} 2 \log _{4} 4=\log _{2} 2=1$,
(ii) $\log _{3}\left(\log _{5} 125\right)=\log _{3} \log _{5} 5^{3}=\log _{3} 3 \log _{5} 5=\log _{3} 3=1$,

Question 16.1 Calculate the logarithm

$$
\text { (i) } \quad \log _{3} 81, \quad \text { (ii) } \quad \log _{7} 16807
$$

Question 16.2 Calculate the value of logarithmic expressions
(i) $\frac{\log _{7} 3125}{\log _{7} 5}$,
(ii) $\frac{\log _{9} 8}{\log _{3} 2}$,
(iii) $\log _{3}\left(\log _{3} \sqrt{7}\right)-\log _{3}\left(\log _{3} 7\right)$,

Question 16.3 Calculate the value of logarithmic expressions
(i) $\quad \log _{5}\left(\log _{5} 3125\right)$,
(ii) $\quad \log _{4}\left(\log _{3} 6561\right)$.

### 16.2 Logarithmic equations and inequalities

An equation in which the unknown occurs under the sign of the logarithm is called the logarithmic equation. We start solving the logarithmic equation by determining its domain. That is, values of the argument $x$ for which the left and right sides of the logarithmic equation have a numeric sense. We are looking for roots in the domain of the logarithmic equation. By solving an equation, we transform the equation into an equation of simple form that may have roots outside its domain. Roots of a logarithmic equation from outside of its domain are called foreign roots. The methods of solving logarithmic equations are based on the properties of the logarithmic and exponential functions. Below we present examples of solving logarithmic equations.

Example 16.4 Solve the equation

$$
\log _{2} x=4
$$

## Solution

First, we define the domain of the logarithmic equation. Namely, the logarithm is only determined for the positive values of $x$. Thus the domain of this equation is the set $x>0$. we write

$$
0<x<\infty \quad \text { or } \quad x \in(0, \infty)
$$

From the definition of the logarithm as the inverse of the exponential function, equality follows

$$
x=2^{4}=16 .
$$

We check that the solution $x=16 \in(0, \infty)$ belongs to the domain of equation

$$
\log _{2} 2^{4}=4 \log _{2} 2=4, \quad \log _{2} 2=1
$$

Example 16.5 Solve the equation

$$
\log _{3}(5-x)+\log _{3}(5+x)=2
$$

## Solution

First, we define the domain of the logarithmic equation. Namely, the logarithm is only determined for the positive values of the argument

$$
5-x>0 \quad \text { and } \quad 5+x>0
$$

Hence the domain of this equation is the set

$$
x<5 \text { or } x>-5 .
$$

Then we write the domain of this equation as an open interval

$$
-5<x<5 \text { or } \quad x \in(-5,5) .
$$

From the property of the sum of logarithms equality follows

$$
\log _{3}(5-x)+\log _{3}(5+x)=\log _{3}(5-x)(5+x)=2 .
$$

By definition, we have an equality

$$
(5-x)(5+x)=3^{2}, \quad \text { or } 25-x^{2}=9 \text { or } x^{2}=16
$$

We calculate the square roots

$$
\sqrt{x^{2}}=|x|, \quad \sqrt{16}=4
$$

Hence we get two solutions

$$
\text { when } \quad|x|=4 \text { to } \quad x_{1}=-4 \quad \text { or } \quad x_{2}=4 .
$$

We check that the solution $x_{1}=-4 \in(-5.5)$ and $x_{2}=4 \in(-5.5)$ belongs to the domain of the equation

$$
\begin{aligned}
& \log _{3}(5+4)+\log _{3}(5-4)=\log _{3} 9 * 1=\log _{3} 3^{2}=2 \\
& \text { and } \\
& \log _{3}(5-4)+\log _{3}(5+4)=\log _{3} 1 * 9=\log _{3} 3^{2}=2
\end{aligned}
$$

Note that both the solutions of $x_{1}=-4 \in(-5.5)$ and $x_{2}=4 \in(-5.5)$ belong to the domain of this equation. Let's mark the domain and the solution on the number line


Number line. The domain of the equation is open interval $(-5,5)$

Example 16.6 Solve the equation

$$
\log _{3}(x--2)+\log _{3}(x-4)=1
$$

## Solution

First, we define the domain of the logarithmic equation. Namely, the logarithm is only determined for the positive values of the argument

$$
x-2>0 \quad \text { and } \quad x-4>0
$$

Hence the domain of this equation consists of $x$ which satisfy inequalities

$$
x>2 \text { or } x>4
$$

Then we write the domain of this equation as the left-hand open endless interval

$$
x>4 \text { or } x \in(4, \infty)
$$

From the property of the sum of logarithms, equality follows

$$
\log _{3}(x-2)+\log _{3}(x-4)=\log _{3}(x-2)(x-4)=1 .
$$

By definition, we have an equality

$$
(x-2)(x-4)=3^{1}, \quad \text { or } x^{2}-6 x+8=3 \text { or } x^{2}-6 x+5=0 .
$$

We calculate the roots of the equation.

$$
x^{2}-6 x+5=0
$$

The discriminant of this equation for the coefficients $a=1, \equiv-6, c=5$ is

$$
\Delta=b^{2}-4 * a * c=6^{2}-4 * 1 * 5=36-20=16 .
$$

Hence, we calculate the roots of the equation

$$
x_{1}=\frac{1}{2}(6-\sqrt{16})=\frac{6-4}{2}=1, \quad x_{2}=\frac{1}{2}(6+\sqrt{16})=\frac{6+4}{2}=5 .
$$

We check that the foreign root $x_{1}=1 \notin(4, \infty)$ does not belong to the domain of the equation, while the root $x_{2}=5 \in(4, \infty)$ belongs to the domain of the equation and the second root $x_{2}=5$ satisfies the equation

$$
\log _{3}(5-2)+\log _{3}(5-4)=\log _{3} 3 * 1=\log _{3} 3=1
$$

Note that only the root of $x_{2}=5 \in(4, \infty)$ belongs to the domain of this equation. Let's mark the domain and the solution on the number line

| -1 | 0 | 1 | 2 | 3 | 4 | $x_{2}=5$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |

Example 16.7 Solve the equation

$$
\log _{2}\left(\log _{4} x\right)=1
$$

## Solution

The domain of this equation is the set of those $x$ for which

$$
\log _{4} x>1, \quad x>4, \quad x \in(4, \infty)
$$

By definition of the logarithm, we know that if

$$
\log _{4} x=2^{1}
$$

then

$$
x=4^{2}, \quad x=16 .
$$

The solution to $x=16 \in(4, \infty)$ belongs to domain.
We check that $x=16$ satisfies the equation

$$
\begin{aligned}
\log _{2}\left(\log _{4} 16\right) & =\log _{2}\left(\log _{4} 4^{2}\right) \\
& =\log _{2}\left(2 \log _{4} 4\right) \\
& =\log _{2} 2=1
\end{aligned}
$$

### 16.2.1 Questions

Question 16.4 Define the domain and find a solution to the equation

$$
\log _{4} x=3
$$

Question 16.5 Define the domain and find a solution to the equation

$$
\log _{4}(1-x)-\log _{4}(1+x)=0
$$

Question 16.6 Solve the equation

$$
\log _{2}(x-1)+\log _{2}(x-2)=1
$$

Question 16.7 Solve the equation

$$
2 \log (x-1)=1
$$

Question 16.8 Solve the equation

$$
\log _{4}\left(\log _{8} x\right)=1
$$

## Chapter 17

## Plane geometry. Planimetry

### 17.1 Introduction

Euclidean geometry, planimetry and stereometry are an important part of mathematics curricula in primary and secondary education.
The scope of planar geometry includes structures with a ruler and a compasses for flat figures as well as measurement relationships between angles and sides in triangles, rectangles, parallelograms, circles and polygons, including regular polygons.

### 17.2 Points, segments and vectors on the plane

Points, lines and planes are primary concepts, they do not need to be defined. A point is understood as a dimensionless geometric figure. A straight line is a one-dimensional Euclidean space that consists of points co-linear. Similarly, a plane creates a Euclidean space composed of points coplanar. Points located on a straight line or on a plane are marked with capital letters $A, B, C, \ldots$; The segment with the beginning at point $A$ and end at point $B$ is denoted by the symbol $[A, B]$. Length of the segment $[A, B]$ at the beginning of $A$ and the end of $B$ is marked with the symbol $|A B|$.
The vector with the package at point $A$ and the end at point $B$ is the segment directed $\overrightarrow{A B}$ with the return from $A$ to $B$. The length of the vector $|\overrightarrow{A B}|$ is equal to is the length of the segment $|A B|$.
line $L$


### 17.3 The position of geometric figures on the plane.

We define the position of geometric figures on the plane in Cartesian coordinates. In the Cartesian coordinate system, the coordinates of points

$$
A=\left(a_{1}, a_{2}\right) \quad \text { and } \quad B=\left(b_{1}, b_{2}\right)
$$

are written in brackets. And the vector

$$
\overrightarrow{A B}=\left[a_{2}-a_{1}, b_{2}-b_{1}\right]
$$

of which the coordinates are equal to the difference of the coordinates of the points $A$ and $B$ is written in square brackets.

Example 17.1 Let the points $A=(3,1.5)$ and $B=(5,3.5)$ form the vector $\overrightarrow{A B}$ with beginning $A$ and ending $B$. Then the vector

$$
\overrightarrow{A B}=[5-3,3.5-1.5]=[2.2]
$$

has the coordinates $x_{1}=2, x_{2}=2$.

## Cartetesian coordinate system



### 17.3.1 Arithmetic operations on points

The sum of two points

$$
A=\left(a_{1}, a_{2}\right) \quad \text { and } \quad B=\left(b_{1}, b_{2}\right)
$$

is equel to the point

$$
P=\left(p_{1}, p_{2}\right)=\left(a_{1}+b_{1}, a_{2}+b_{2}\right)
$$

with coordinates $p_{1}=a_{1}+b_{1} \quad$ and $\quad p_{2}=a_{2}+b_{2}$.
Example 17.2 Calculate the sum of points

$$
A=(1,2) \quad \text { and } \quad B=(2,1)
$$

Solution. The sum

$$
A+B=(1.2)+(2.1)=(1+2.2+1)=(3.3)
$$

Answer. The sum of the given points $A=(1,2)$ and $B=(2,1)$ is the third point $\quad P=(3,3)$. with the coordinates $x_{1}=3, x_{2}=3$.
Below in the figure is the geometric interpretation of the sum of points

$$
A+B=(1,2)+(2,1)
$$



Difference of two points

$$
A=\left(a_{1}, a_{2}\right) \quad i \quad B=\left(b_{1}, b_{2}\right)
$$

is equal to the point $P=\left(p_{1}, p_{2}\right)=\left(a_{1}-b_{1}, a_{2}-b_{2}\right)$
for coordinates $p_{1}=a_{1}-b_{1} \quad i \quad p_{2}=a_{2}-b_{2}$.
Example 17.3 Calculate the difference of the points

$$
A=(1,2) \quad i \quad B=(2,1)
$$

Solution. The difference

$$
A-B=(3.3)-(2.1)=(3-2.3-1)=(1.2)
$$

Answer. The difference between the given points $A=(3,3)$ and $B=(2,1)$ is the point $P=(1,2)$.


### 17.3.2 Vectors on the plane

Let points $A=\left(a_{1}, a_{2}\right)$ and $B=\left(b_{1}, b_{2}\right)$ begiven.
The vector $\overrightarrow{A B}$ about the package at the point $A=\left(a_{1}, a_{2}\right)$ and the end at the point $B=\left(b_{1}, b_{2}\right)$ we define as the difference of points

$$
\overrightarrow{A B}=B-A=\left[b_{1}-a_{1}, b_{2}-a_{2}\right] .
$$

${ }^{2}{ }^{3}$ For example, a bound vector beginning at $A=(0,1)$ and ending at $B=(2,0)$ has the coordinates

$$
\overrightarrow{A B}=b-a=(2,0)-(0,1)=[2,-1] .
$$

### 17.3.3 Arithmetic operations on vectors

## Sum of vectors

The sum of two vectors

$$
\vec{v}=\left[v_{1}, v_{2}\right] \quad \text { and } \quad \vec{w}=\left[w_{1}, w_{2}\right]
$$

is equal to the vector

$$
\vec{Q}=\left[z_{1}, z_{2}\right]=\left[v_{1}+w_{1}, v_{2}+w_{2}\right]
$$

with coordinates

$$
z_{1}=v_{1}+w_{1} \quad \text { and } \quad z_{2}=v_{2}+w_{2}
$$

Example 17.4 Calculate the sum of vectors

$$
\vec{v}=[1,2] \quad \text { and } \quad \vec{w}=[2,1]
$$

[^14]Solution. We find

$$
\vec{v}+\vec{w}=[1,2]+(2,1)=[1+2,2+1]=[3,3]
$$

Answer. The sum of the given points $\vec{v}=[1,2]$ i $\vec{w}=[2,1]$ is the vector $\vec{Q}=[3,3]$.


Difference of vectors The difference of the two vectors $\vec{v}=\left[v_{1}, v_{2}\right] \quad i \quad \vec{w}=\left[w_{1}, w_{2}\right]$ is equal to vector

$$
\vec{Q}=\left[z_{1}, z_{2}\right]=\vec{v}-\vec{w}=\left[v_{1}-w_{1}, v_{2}-w_{2}\right]
$$

with the coordinates $z_{1}=v_{1}-w_{1} \quad i \quad z_{2}=v_{2}-w_{2}$.
Example 17.5 Calculate the difference of vectors $\vec{v}=[1,2] \quad i \quad \vec{w}=[2,1]$
Solution. We calculate the difference of vectors

$$
\vec{v}-\vec{w}=[1,2]-[2,1]=[1-2,2-1]=[-1,1]
$$

Answer. The result of subtracting the data of vectors $\vec{v}=[1,2]$ i $\vec{w}=[2,1]$ is the vector $\vec{Q}=$ $[-1,1]$.


### 17.3.4 Scalar product of vectors

${ }^{4}$ The scalar product of vectors is important operation on vectors that is used in mathematics, physics, chemistry, and other science.

Definicja 17.1 The scalar product of the vectors $\vec{v}=\left[v_{1}, v_{2}\right] i \vec{w}=\left[w_{1}, w_{2}\right]$ is the number defined by the following formula

$$
\begin{equation*}
(\vec{v}, \vec{w})=v_{1} * w_{1}+v_{2} * w_{2} \tag{17.1}
\end{equation*}
$$

The value of the scalar product of vectors is not a vector, but it is a number
Example 17.6 Calculate the scalar product of vectors

$$
\vec{v}=[2,5] \quad \text { and } \quad \vec{w}=[7,3] .
$$

[^15]Solution. Using the formula (17.1) we calculate the scalar product of the given vectors, we write

$$
(\vec{v}, \vec{w})=([2,5] *[7,3])=2 * 7+5 * 3=14+15=29
$$

Answer. The value of the scalar product of the given vectors $\vec{v}=[2.5]$ i $\vec{w}=[7.3]$ is a number 29, we write

$$
(\vec{v}, \vec{w})=29 .
$$

The scalar product of vectors preserves all the properties of the multiplication operation. Let's consider two vectors

$$
\vec{v}=\left[v_{1}, v_{2}\right] \quad \text { and } \quad \vec{w}=\left[w_{1}, w_{2}\right]
$$

- scalar product is commutative

$$
(\vec{v}, \vec{w})=(\vec{w}, \vec{v})
$$

Indeed, we check that

$$
(\vec{v}, \vec{w})=v_{1} * w_{1}+v_{2} * w_{2}=w_{1} * v_{1}+w_{2} * v_{2}=(\vec{w}, \vec{v})
$$

- Scalar product of vectors is separable with respect to the vector addition operation

$$
(\vec{v},(\vec{w}+\vec{Q}))=(\vec{v}, \vec{w})+(\vec{v}, \vec{Q})
$$

Indeed we check that

$$
\begin{aligned}
(\vec{v}, \vec{w}+\vec{Q}) & =v_{1} *\left(w_{1}+z_{1}\right)+v_{2} *\left(w_{2}+z_{2}\right) \\
& =v_{1} * w_{1}+v_{1} * z_{1}+v_{2} * w+2+v_{2} * z_{2} \\
& =\underbrace{v_{1} * w_{1}+v_{2} * w_{2}}_{(\vec{v}, \vec{w}}+\underbrace{v_{1} * z_{1}+v_{2} * z_{2}}_{(\vec{v}, \vec{Q})} \\
& =(\vec{v}, \vec{w})+(\vec{v}, \vec{Q})
\end{aligned}
$$

- scalar product of the vector $\vec{v}$ by itself is equal to the square of its length

$$
(\vec{v}, \vec{v})=v_{1} * v_{1}+v_{2} * v_{2}=v_{1}^{2}+v_{2}^{2}=|\vec{v}|^{2}
$$

The following theorem holds:
Theorem 17.1 Vectors $\vec{v}$ and $\vec{w}$ are perpendicular if and only if their scalar product is equal to zero. We write it in the formula

$$
\vec{v} \perp \vec{w} \Longleftrightarrow(\vec{v}, \vec{w})=0 .
$$

Proof. There are a few proofsof this claim. Here we will give a proof based on the Pythagorean theorem. Namely, we will prove that the triangle with the arms $\vec{v}$ and $\vec{w}$ is rectangular if and only if the scalar product is equel to zero

$$
(\vec{v}, \vec{w})=0
$$

We calculate the square of the length of the vector difference $\vec{v}$ i $\vec{w}$

$$
\begin{aligned}
|\vec{v}-\vec{w}|^{2} & =(\vec{v}-\vec{w}, \vec{v}-\vec{w}) \\
& =(\vec{v}, \vec{v})-2(\vec{v}, \vec{w})+(\vec{w}, \vec{w}) \\
& =|\vec{v}|^{2}-2(\vec{v}, \vec{w})+|\vec{w}|^{2}
\end{aligned}
$$

Note that if the scalar product is equal to zero

$$
(\vec{v}, \vec{w})=0
$$

then the sides of the triangle $\triangle A B C$

$$
|A B|=|\vec{v}|, \quad|A C|=|\vec{w}|, \quad|B C|=|\vec{v}-\vec{w}|
$$

satisfy the equality

$$
\begin{equation*}
|\vec{v}|^{2}+|\vec{w}|^{2}=|v \overrightarrow{-} w|^{2} \tag{17.2}
\end{equation*}
$$



On the other hand, the sum of the squares of the two sides of a triangle is equal to the square of the length of the third side if and only if the triangle is rectangular.
Thus, the angle $\angle A C B$ between the vectors $\vec{v}$ and $\vec{w}$ is right angle if and only if the scalar product of these vectors is equal to zero.
End of proof.

Example 17.7 Calculate the scalar product and the length of the vectors

$$
\vec{v}=[6,8], \quad \vec{w}=[9,12] .
$$

Solution. We calculate the scalar product using the formula (17.1) for the vectors

$$
\vec{v}=\left[v_{1}, v_{2}\right]=[6,8], \quad \text { and } \quad \vec{w}=\left[w_{1}, w_{2}\right]=[9,12]
$$

So the scalar product

$$
(\vec{v}, \vec{w})=6 * 9+8 * 12=54+96=150
$$

is equal 100 .
We know that the square of the length of the vector $\vec{v}=[6.8]$ is equal to the scalar product of this vector through itself.

$$
|\vec{v}|^{2}=(\vec{v}, \vec{v})=6 * 6+8 * 8=36+64=100
$$

Where does the length of the vector come from?

$$
|\vec{v}|=\sqrt{100}=10
$$

Similarly, we calculate the length of the vector $\vec{w}=[9,12]$

$$
|\vec{w}|=\sqrt{(\vec{w}, \vec{w})}=\sqrt{9 * 9+12 * 12}=\sqrt{81+144}=\sqrt{225}=15 .
$$

Example 17.8 For what value of paramiter $m$ the vectors

$$
\vec{v}=[m, 6], \quad \vec{w}=[3,2] .
$$

are perpendicular?
Solution. We calculate the scalar product using the formula (17.1) for the vectors

$$
\vec{v}=\left[v_{1}, v_{2}\right]=[m, 6], \quad \text { and } \quad \vec{w}=\left[w_{1}, w_{2}\right]=[3,2]
$$

Vectors are orthogonal if their scalar product is zero. We calculate the scalar product

$$
(\vec{v}, \vec{w})=m * 3+6 * 2=3 m+12=0 .
$$

Hence the scalar product is zero

$$
6 m+12=0, \quad \text { dla } \quad m=-\frac{12}{3}=-4 .
$$

In fact, we check that for $m=-4$ the scalar product of the vector $\vec{v}=[m, 6]$ by the vector $\vec{w}=[3,2]$ is equal to zero

$$
(\vec{v}, \vec{w})=-4 * 3+6 * 2=0
$$

Answer.: Vectors

$$
\vec{v}=\left[v_{1}, v_{2}\right]=[m, 6], \quad \text { and } \quad \vec{w}=\left[w_{1}, w_{2}\right]=[3,2]
$$

are perpendicular for the parameter $m=-4$.
Question 17.1 Calculate the scalar product and the length of the vectors

$$
\vec{v}=[12,16], \quad \vec{w}=[15,20] .
$$

Question 17.2 For what value of the parameter $m$ vectors

$$
\vec{v}=[m, 15], \quad \vec{w}=[5,3] .
$$

are perpendicular?

### 17.4 Constructs with compasses and ruler

The basic constructions using a compasses and a ruler include

- Symmetrical division of a segment
- Construction of a line perpendicular to a given straight line that passes through a given point lying beyond the straight line,
- Construction of the bisector of a given angle,
- Constructure of straight parallel lines,
- Construction of a triangle with given sides,
- Construction of a quadrilateral with given sides.


### 17.4.1 Symmetrical division of a segment.

Let the segment $[A, B]$ be given with the length $a=|A B|$. Place the compasses at the point $A$ and with the compasses width greater than half of the segment $[A, B]$ we draw two arcs above and below the episode. We draw a line $L$ through the points that intersect the arcs. The straight line $L$ is the symmetrical line of the segment $[A, B]$.

Question 17.3 Ddraw a symmetrical line of given segment $[A, B]$ of length 6 cm using a compasses and a ruler.

### 17.4.2 Construction of a line perpendicular to a given straight line

Consider the line $L$ and the point $P$ not lying on $L$. We place the compasses at the point $P$ and draw an arc intersecting straight line $L$ in points $A$ and $B$. Place the compasses at the point $A$ and draw an arc. Then we place the compasses at $B$ and draw an arc. We mark the point of intersection of the arcs with the letter $P^{*}$. Use the points $P^{*}$ and $P$ to draw a straight line $L^{*}$, as in the figure below.


Question 17.4 Draw the line perpendidular to the given line $L$ through a given point $P$ not lying on the line L, using a compasses and a ruler.

### 17.4.3 Construction of bisector of an angle

Let $\alpha$ be a given angle with vertex at point $O$ and arms $L_{1}$ and $L_{2}$.


In order to draw besection of the angle $\alpha$ we place the compass at the vertex $O$. Next we draw an arc crossing the arms of $L_{1}$ and $L_{2}$ in points $A$ and $B$. The points $A$ and $B$ are equally distant from the point $O$, we write $|O A|=|O B|$. Then we place the compasses at the point $A$ and circle łuk. Similarly, we place the compasses at $B$ and circle the arc. We denote the point of intersection of the arcs with the letter $P$. Through the points $O$ and $P$ we run a bisector of the angle


Question 17.5 Draw the bisector of the angle $\alpha$ given below using a compasses and a ruler.


Construction of a line parallel to a given line $L$. The construction of a line parallel to a given line $L$ and passing through a given point $P$ is based on drawing a parallelogram.
$\cdot P$

L

In the first step of the construction, we place the compasses at the given point $P$ and we draw the arc that intersects the given straight line $L$ in two points $A$ and $B$ as below on the plot ${ }^{\bullet} P$


In the second step of the construction, we connect the point $A$ intersection with the given point $P$ with a ruler. Then we place the compasses at the point $B$ and we draw an arc with the compasses width equal to the distance $|A P|$ of the $A$ point from the $P$ point.


In the third step of the construction, we place the compasses at the point $P$ and with the compasses width equal to the distance $|A B|$ of the $A$ point from the $B$ point, we draw the second arc. We
mark the point of intersection of the arcs with the letter $P^{*}$.


In the fourth step of the construction, we draw with a straight ruler through the points $P$ and $P^{*}$. Finally, we connect the points $B$ and $P^{*}$ with the ruler.


We can see that in this way we have drawn a parallelogram $A B P P^{*}$, whose side $\left[P, P^{*}\right]$ lies on the line $L^{*}$ parallel to the line $L$ passing through the given point $P$.

Question 17.6 Draw a line parallel to the line in the picture and passing through the given point

### 17.4.4 Two parallel lines cut by a third straight line

Let us consider two parallel lines $L_{1}$ and $L_{2}$ intersected by the third line $L$. Below in the graph we observe the pairwise equal angles.


Two parallel straight lines $L_{1}$ and $L_{2}$ are intersected by a third straight line $L$

- pairwise vertex angles equal

$$
\angle 1=\angle 4, \quad \angle 2=\angle 3, \quad \angle 5=\angle 8, \quad \angle 6=\angle 7
$$

- pairwise angles equal

$$
\angle 1=\angle 5, \quad \angle 3=\angle 7, \quad \angle 2=\angle 6, \quad \angle 4=\angle 8
$$

- angles alternating inner pairs equal

$$
\angle 3=\angle 6, \quad \angle 4=\angle 5,
$$

- alternating outer angles in pairs equal

$$
\angle 1=\angle 8, \quad \angle 2=\angle 7,
$$

- adjacent angles, the sum of which is equal to $180^{\circ}$

$$
\begin{array}{ll}
\angle 1 \text { adjacent to } \angle 2, & \angle 3 \text { adjacent to } \angle 4, \quad \angle 1 \text { adjacent to } \angle 3, \\
\angle 2 \text { adjacent to } \angle 4, & \angle 5 \text { adjacent to } \angle 6, \angle 7 \text { adjacent to } \angle 8 \text {, } \\
\angle 5 \text { adjacent to } \angle 7, & \angle 6 \text { adjacent to } \angle 8
\end{array}
$$

Question 17.7 One of the vertex angles is equal to $30^{\circ}$.


Two parallel straight lines cut by a third straight line

Calculate all angles
(a) vertex angles
(b) locally alternating angles
(c) corresponding angles
(d) adjacent interior angles
(e) adjacent outer

Mark the values of all angles in the graph

### 17.5 Circle

The area inside the circle is also called a circle.
The perimeter of the circle

$$
O_{\text {perimiter }}=2 * \pi * r
$$

circle field

$$
P_{\text {circle }}=\pi * r^{2}, \quad \pi \approx \frac{314}{100}=3.14
$$


the diameter of the circle is 2 times the radius of the circle.
Question 17.8 Draw a circle with a compasses with a radius of 3 cm . Use a crayon to mark the inside of the circle of 3 cm radius.
Calculate the diameter of the circle, the circumference of the circle, the area of the circle.

### 17.5.1 Arc angle measure

Let us consider a circle with a radius of $R$


The arc measure of the angle $\alpha=\angle B C A$ based on the arc $l=\widehat{A B}$ is defined as the ratio of the length of the arc $l$ to the radius of $R$

$$
\alpha=\frac{l}{R}
$$

The full angle, which is $360^{\circ}$ in an angular measure, is based on the arc

$$
l=2 \pi * R
$$

equal to the circumference of the circle.
So the arc measure of the full angle is equal to

$$
\alpha=\frac{2 \pi * R}{R}=2 \pi
$$

Likewise, a semi-full angle that is $180^{\circ}$ is based on an arc

$$
l=\pi * R
$$

is equal to half the circumference of the circle. This means that the arc measure of a half angle is equal to

$$
\alpha=\frac{\pi * R}{R}=\pi
$$

Also, a right angle that is $90^{\circ}$ on an angle is based on an arc

$$
l=\frac{2 \pi * R}{4}=\frac{\pi * R}{2}
$$

equal to the fourth part of the circumference of the circle. This means that the arc measure of a right angle is equal to

$$
\alpha=\frac{\pi * R}{2 R}=\frac{\pi}{2}
$$

In fact, the arc measurement arc does not depend on the length of the radius $R$. Therefore, we can take the radius of the circle $R=1$.
The unit of measure for the arc angle is 1 radian. The full angle has $2 * \pi$ radians, which is equal to $360^{\circ}$ in angular terms. So, one degree

$$
1^{0}=\frac{2 * \pi}{360}=\frac{\pi}{180} \text { radianow }
$$

and

$$
1 \text { radian }=\frac{180^{0}}{\pi} \text { stopni }
$$

Example 17.1 Find the arc measure of angle $30^{\circ}$.
Solution. We use proportion, angle $180^{\circ}$ corresponds to the arc measure of this angle $\pi$ radians. Thus, the angle $30^{\circ}$ corresponds to the arc measure $x$ radians. We write this proportion with the equation

$$
\frac{\pi}{180}=\frac{x}{30}
$$

Where do we calculate the arc measure of an angle $30^{\circ}$

$$
x=\frac{30 * \pi}{180}=\frac{\pi}{6}
$$

Question 17.9 Find the arc measure of angle $\alpha$, if its angular measure is equal to

$$
\begin{equation*}
\text { (i) } \quad \alpha=30^{\circ} \quad \text { (ii) } \quad \alpha=60^{\circ} \quad \text { (iii) } \quad \alpha=120^{\circ} \tag{iii}
\end{equation*}
$$

Question 17.10 How many degrees is the angle $\alpha$, if its arc measure is equal to
(i) $\quad \alpha=\frac{3 \pi}{4}$
(ii) $\quad \alpha=\frac{5 \pi}{4}$
(iii) $\quad \alpha=\frac{4 \pi}{3}$
(iv) $\alpha=\frac{5 \pi}{6}$

### 17.5.2 Angle inscribed in circle and the central angle

The angle inscribed in the circle with the radius $R$ and the center at the point O is the angle $\alpha$ whose vertex $C$ lies on the circle and the arms $A C$ and $B C$ intersect the circle at points $A$ and $B$ as below on the graph


From the definition of the arc measure and the angular measure, we know that the value of the inscribed angle $\angle B C A=\alpha$ does not depend on the size of the radius $R$. So we assume that the radius $R=1$.

Lemma 17.1 The value of the angle $\angle B C A=\alpha$ inscribed in the circle is constant and independent of the position of the vertex $C$ and the radius $R$ of the circle centered in $O$.


The change of position of the angle $\angle B C A=\alpha$ inscribed in the circle to position $C^{*}=\angle B C^{*} A=\alpha$, does not change the $\alpha$ value of the embedded angle in the circle.
Indeed, the angle $\alpha$ with at the vertex at $C$ and with the arms $A C$ and $B C$ which intersect the circle at points $A$ and $B$. The value of angle $\alpha$ in the radians $l=\widehat{A B}$ coresponds to degrees

$$
\alpha=\frac{l * \pi}{180^{0}} .
$$

If the vertex $C$ moves in the direction of the point $A$, then the entered angle $\alpha=\frac{l * \pi}{180^{0}}$, does not change value, because the length of the $\operatorname{arc} l=\widehat{A B}$ remains the same.
If the vertex $C$ coincides with the point $A$, then the interval $A C$ will reduce to the point $A$. If the vertex $C$ exceeds the point $A$ and continues to move towards the point $B$ then the angle $\alpha$ will have an arc of length $2 \pi-l$, and its angular measure

$$
\alpha=\frac{(2 \pi-l) * 180^{\circ}}{\pi} .
$$

Center angle. The central angle is the angle between the radii of the circle $R=|A O|$ and $R=|B O|$ at the vertex in the center of the circle $O$.


### 17.5.3 Relationship between center angle and inscribed angle

The following theorem holds
Theorem 17.2 The center angle, based on the same arc as the inscribed angle, is twice the inscribed angle. Thus the equality holds


We give two proofs of the inscribed angle and central angle theorem.
Proof 1. From Lemma 1, we know that the value of the entered angle $\alpha$ does not depend on the position of its vertex $C$ on the circle.
Therefore we assune the position of the vertex $C$ on the diameter of the circle passing through the vertex $C$ and the center of the circle $O$.
The radii of the circle $A O, B O$ and $C O$ form triangles $\triangle A O C$ i $\triangle B C O$ equal and congruent.
So their angles $\beta, \gamma$ and the center angle $\angle B O A=\lambda$ satisfy the equations

$$
\begin{aligned}
& \alpha=2 \beta \\
& \gamma+2 \beta=\pi \\
& 2 \gamma+\lambda=2 \pi .
\end{aligned}
$$

Where do we calculate the center angle

$$
\lambda=2 \pi-2 \gamma=2 \pi-2 \underbrace{(\pi-2 \beta)}_{\gamma}=4 \beta=2 \alpha .
$$

Proof 2. First, we will provide the proof in the case when the center of the circle $O$ lies between the arms of $A C$ and $B C$ of the inscribed angle $\angle B C A=\alpha$, then in the case when the center of the circle $O$ lies outside the arms of the inscribed angle.
In the first case, we notice that isosceles triangles $\triangle A O C, \triangle B C O$ with arms equal to the radius of the circle $R$ have angles at their bases equal. The triangle $\triangle A O C$ has angles equal to $\delta$ at the base of $A C$ and the triangle $\triangle B C O$ has angles equal to $\beta$ atthebaseof BC . Inserted angle $\angle B C A=\alpha$ is marked with the Greek letter $\alpha$, and the center angle $\angle A O B$ is marked with the Greek letter $\lambda$, as in the picture.
Then we notice that the angles $\alpha, \beta, \gamma, \delta, \epsilon, \lambda$ satisfy the system of linear equations

$$
\begin{array}{ll}
\alpha=\beta+\delta, & \angle B C A \text { inscibed } \alpha \\
2 \beta+\gamma=\pi, & \text { sum angles of atriangle } \triangle B C O \text { rowna } \pi \\
2 \delta+\epsilon=\pi, & \text { sum angles of atriangle } \triangle A O C \text { rowna } \pi  \tag{17.3}\\
\gamma+\epsilon+\lambda=2 \pi, & \text { theangle full is equalto } 2 \pi
\end{array}
$$

We will solve the system of linear equations (17.3) using the substitution method. Namely, from the equation of the second and third in the system (17.3) we calculate

$$
\begin{aligned}
& \gamma=\pi-2 \beta, \\
& \epsilon=\pi-2 \delta
\end{aligned}
$$

Hence sum of angles

$$
\gamma+\epsilon=2 \pi-2(\beta+\delta)
$$

From the fourth equation in the system of equations (17.3) we calculate the median angle

$$
\lambda=2 \pi-(\gamma+\epsilon)=2 \pi-\underbrace{(2 \pi-2(\beta+\delta))}_{\gamma+\epsilon}=2 \underbrace{(\beta+\delta)}_{\alpha}=2 \alpha
$$

So, we figured out that the central angle $\lambda$ is twice the angle entered $\alpha$, we write below

$$
\lambda=2 \alpha
$$

End of proof of the first case.
Proof. Proof in the second case, when the center of the circle $O$ lies outside the arms of $A C$ and $B C$ of the inserted angle $\angle B C A=\alpha$.


We note that the isosceles triangles $\triangle A O C$ and $\triangle B O C$ with arms equal to the radius of the circle $R$ have angles equal, respectively

$$
\angle O C A=\angle C A O=\alpha+\beta \quad \text { and } \quad \angle C B O=\angle O C B=\beta .
$$

The sum of the angles in the triangles $\triangle A O C$ i $B O C$ is equal to $180^{\circ}$ or $\pi$, we write

$$
\begin{array}{ll}
\underbrace{\angle O C A+\angle C A O}_{2(\alpha+\beta)}+\underbrace{\angle A O C}_{\delta}=\pi & 2(\alpha+\beta)+\delta=\pi \\
\underbrace{\angle O B C+\angle O C B}_{2 \beta}+\underbrace{\angle B O C}_{\delta+\lambda}=\pi & 2 \beta+\delta+\lambda=\pi \tag{17.4}
\end{array}
$$

The system of linear equations (17.4) will be solved by the substitution method.
Namely, we calculate from the first equation

$$
\delta=\pi-2(\alpha+\beta)
$$

and substitute into the second equation

$$
2 \beta+(\pi-2(\alpha+\beta))+\lambda=\pi, \quad \lambda-2 \alpha=0 .
$$

From where the central angle $\angle B O A=2 \alpha$ is twice the angle entered $\angle O C A=\alpha$, we write

$$
\lambda=2 \alpha
$$

or

$$
\angle B O A=2 \angle O C A
$$

Koniec dowodu przypadku 2.
Conclusion:The angle based on the diameter of the circle is straight, has $90^{\circ}$, the arc measure is equal $\frac{\pi}{2}$ radians.

### 17.6 Triangles

### 17.6.1 Construction of a triangle with given sides

Let us consider three segments $[A, B],[B, C]$, and $[A, C]$
$A \longrightarrow B$
B $\qquad$ C

A $\qquad$ C
Let us choose the segment $[A B]$ as the base of the triangle $\triangle A B C$. With the compasses width equal to the length of segment $[A, C]$ we draw an arc, placing the compasses at point $A$. Then with the compasses width equal to the length of segment $[B, C]$ we draw an arc by placing a compasses at point $B$. The point $C^{*}$ of intersection of arcs, we connect with points $A$ and $B$. ${ }^{5}$


Let us note that a triangle can only be built when these segments satisfy triangle inequality The sum of the lengths of two sides of the triangle is greater than the length of third side. Indeed, we have the following relations

$$
\begin{aligned}
& |A B|+|B C| \geq|A C|, \\
& |A B|+|A C| \geq|B C|, \\
& |A C|+|B C| \geq|A B|
\end{aligned}
$$

### 17.6.2 The sum of angles of a triangle

The sum of the angles of each triangle equals $180^{\circ}$, or in radians $\pi$. Below on the graph the geometrical interpretation of sum of angles of a triangle.

[^16]

From the figure, we can see that the sum of the angles of each triangle is $180^{\circ}$. Indeed, the line $D C$ is parallel to the base $A B$ of the triangle $A B C$. Internal alternating angles $\alpha$ at base and $\alpha$ at $D C$ are equal. Also angles $\beta$ at base $A B$ and $\eta$ at $D C$ are equal.
Thus we can see that

$$
\alpha+\beta+\gamma=180^{\circ}
$$

This means that the sum of the angles of each triangle is $180^{\circ}$.

### 17.6.3 Equilateral triangle.

An equilateral triangle has all sides equal and all angles equal $\alpha=60^{\circ}$, in radians $\alpha=\frac{\pi}{3}$ as in the picture

Constructing an equilateral triangle. We draw a segment with a fixed length of the sides of the triangle. We put the compasses foot at the beginning of the segment and draw a circle with a radius equal to the segment's length. Then we place the compasses foot at the other end of the segment and draw a circle with the same rad of the circles with the ends of the segment. In this way, a triangle with equal sides and equal angles was created.


Equilateral triangle $\triangle A B C$
The height $h$ of the triangle $\Delta A B C$ is a bisector of the angle $\alpha$ and divise the base $a$ at point $D$ on half. Similarly, the heights of an equilateral triangle on the other sides divide them into two halves and all intersect at one point $O$. The point $O$ intercept divides these hights in propotion 1:3.
Thus the following proportion holds

$$
\frac{|D O|}{|D C|}=\frac{1}{3}, \quad|D C|=h
$$

Indeed, we have

$$
|D O|=\frac{1}{3} h, \quad \text { i } \quad|O C|=\frac{2}{3} h
$$

From the Pythagorean theorem we calculate the value of hight $h$ of triangle $\triangle A B C$

$$
h^{2}=a^{2}-\left(\frac{a}{2}\right)^{2}=\frac{3}{4} a^{2},
$$

Hence the height of an equilateral triangle

$$
h=\frac{a \sqrt{3}}{2}
$$

Now we can calculate the area of an equilateral triangle with sides equal $a$

$$
\begin{aligned}
& P=h * \frac{a}{2}=\frac{a \sqrt{3}}{2} * \frac{a}{2}=\frac{a^{2} \sqrt{3}}{4} \\
& P=\frac{a^{2} \sqrt{3}}{4}
\end{aligned}
$$

Question 17.11 Measure the sides and angles of the triangle $\triangle A B C$ below in the draw


Calculate the area and perimeter of the triangle $\triangle A B C$, with side $a=3 \mathrm{~cm}$.

### 17.6.4 Isosceles triangle

An isosceles triangle with the base equal to the segment $[A, B]$ and equal arms $[A, C]=[B, C]$ has angles equal to $\angle$ atthebase $[\mathrm{A}, \mathrm{B}] C A B=\angle A B C=\alpha$. Height $h=|C D|$ divides the base $[A, B]$ in half.
The angle at the vertex of $C$ in an angular measure

$$
\beta=180^{\circ}-2 * \alpha
$$

or arc measure

$$
\beta=\pi-2 * \alpha
$$

The height $h=|C D|$ divides the base of $[A b]$ in half


The area of an isosceles triangle $\triangle A B C$ is calculated using the general formula

$$
P_{\triangle A B C}=\frac{1}{2} \underbrace{|A B| *|C D|}_{a * h}=\frac{1}{2} a * h .
$$

Question 17.12 Consider the isosceles triangle $\triangle A B C$. Calculate the perimeter, area and sum of angles of an isosceles triangle when length of base equals $|A B|=3 \mathrm{~cm}$, and length of arms equal $|A C|=|B C|=4 \mathrm{~cm}$.

### 17.6.5 Rectangular triangle

In a right triangle, we distinguish the right angle $\angle C A B=\alpha$ with vertix at point A and two angles $\angle A B C=\beta$ and $\angle B C A=\gamma$. The legs $[A, C]=b$ and $[A, B]=c$ adjacient to right angle $\alpha=90^{\circ}$. Hypotenuse $[B, C]=a$ lying opposite to right angle. ${ }^{6}$


Rectangular triangl $\triangle A B C$
Area of the triangle $=\frac{a * b}{2}$, perimeter of the triangle $=a+b+c$
Question 17.13 Draw a triangle with the same angles and sides twice as short, using a ruler and a compasses. Calculate the sum of the angles of this triangle

### 17.7 Congruent triangles

Two triangles are congruent if their respective sides and angles are equal. Obviously, two triangles are congruent, if one of the three listed below qualities of congruence triangles holds.
First features of triangle congruence.
Two triangles $\Delta A B C$ and $\Delta A^{\prime} B^{\prime} C^{\prime}$ are congruent if their respective sides are equal.

$\alpha=\alpha^{\prime}$

$$
\beta=\beta^{\prime}
$$

$$
\gamma=\gamma^{\prime}
$$

$a=a^{\prime}$
$b=b^{\prime}$
$c=c^{\prime}$

$$
[A, B]=\left[A^{\prime}, B^{\prime}\right]
$$

$[A, C]=\left[A^{\prime}, C^{\prime}\right]$,
$[B, C]=\left[B^{\prime \prime} C^{\prime}\right]$
Second feature of triangle congruence. Two triangles are congruent if they have one of sides coresponding equal and the angles adjacent to that side also equal

$$
\begin{array}{cl}
{[A, B]=\left[A^{\prime}, B^{\prime}\right],} & c=c^{\prime} \\
\alpha=\alpha^{\prime} & \beta=\beta^{\prime}
\end{array}
$$

Third feature of triangle congruence. Two triangles are congruent if they have two sides equal and the angles between them also equal.

[^17]\[

$$
\begin{array}{rlrlrl}
{[A, B]} & =\left[A^{\prime}, B^{\prime}\right], & {[A, C]} & =\left[A^{\prime}, C^{\prime}\right] & & \\
c & =c^{\prime}, & b & =b^{\prime} & \alpha=\alpha^{\prime}
\end{array}
$$
\]

### 17.8 Similarity of triangles

The two triangles $\triangle A B C$ and $A^{\prime} B^{\prime} C^{\prime}$ are similar, we write

$$
\Delta A B C \simeq \Delta A^{\prime} B^{\prime} C^{\prime}
$$

if they have corresponding sides proportional in the $k$ ratio, that is

$$
\begin{aligned}
& \frac{|A B|}{\left|A^{\prime} B^{\prime}\right|}=\frac{|A C|}{\left|A^{\prime} C^{\prime}\right|}=\frac{|B C|}{\left|B^{\prime} C^{\prime}\right|}=k \\
& |A B|=k *\left|A^{\prime} B^{\prime}\right| \\
& |A C|=k *\left|A^{\prime} C^{\prime}\right| \\
& |B C|=k *\left|B^{\prime} C^{\prime}\right|
\end{aligned}
$$

Let us consider the triangle $\triangle A B C$ with sides

$$
|A B|=c=4 c m, \quad|B C|=c=2 c m, \quad|A C|=b=3.5 \mathrm{~cm}
$$

and the triangle $\Delta A^{\prime} B^{\prime} C^{\prime}$ with sides

$$
\left|A^{\prime} B^{\prime}\right|=c^{\prime}=8 \mathrm{~cm}, \quad\left|B^{\prime} C^{\prime}\right|=c^{\prime}=4 \mathrm{~cm}, \quad\left|A^{\prime} C^{\prime}\right|=b=7 \mathrm{~cm}
$$

Note that the proportions of the triangles $\triangle A B C i \Delta A^{\prime} B^{\prime} C^{\prime}$ is equal to $k=2$.


Below we give three features of the similarity of triangles.
The first feature similarity of triangles. The two triangles $\triangle A B C$ i $\Delta A^{\prime} B^{\prime} C^{\prime}$ are similar if they have sides proportional in the proportion scale $k$, that is

$$
\begin{aligned}
& |A B|=k *\left|A^{\prime} B^{\prime}\right|, \\
& |A C|=k *\left|A^{\prime} C^{\prime}\right|, \\
& |B C|=k *\left|B^{\prime} C^{\prime}\right| .
\end{aligned}
$$

Question 17.14 Draw the triangle $\triangle A B C$ with sides

$$
|A B|=c=5 \mathrm{~cm}, \quad|A C|=b=4 c m, \quad|B C|=a=3 \mathrm{~cm}
$$

using a ruler and a compasses.
Draw a second triangle $\Delta A^{\prime} B^{\prime} C^{\prime}$ with sides twice the longer of sides triangle $\triangle A B C$.

The second feature of the similarity of triangles. Two triangles $\triangle A B C$ i $\Delta A^{\prime} B^{\prime} C^{\prime}$ are similar if they have two proportional sides in the ratio scale $k$ and the angles between these sides equal, that is $\alpha=\alpha^{\prime}$

$$
\begin{aligned}
& \frac{|A B|}{\left|A^{\prime} B^{\prime}\right|}=\frac{|A C|}{\left|A^{\prime} C^{\prime}\right|}=k, \\
& |A B|=k *\left|A^{\prime} B^{\prime}\right| \\
& |A C|=k *\left|A^{\prime} C^{\prime}\right|
\end{aligned}
$$

Question 17.15 Draw a triangle $\triangle A B C$ with sides

$$
|A B|=c=3 \mathrm{~cm}, \quad|A C|=b=5 \mathrm{~cm},
$$

and given angle $\alpha=45^{\circ}$

using a ruler and a compasses.
Draw a second triangle $\Delta A^{\prime} B^{\prime} C^{\prime}$ with sides twice the longer then sides of the triangle $\triangle A B C$.
The third feature of similarity of triangles. The two triangles $\triangle A B C$ and $\Delta A^{\prime} B^{\prime} C^{\prime}$ are similar if they have the sides $A B$ and $A^{\prime} B^{\prime}$ proportional in the scale of $k$ and the adjacent angles are equal, that is, $\alpha=\alpha^{\prime}$ i $\beta=\beta^{\prime}$ and $A B$ and $A^{\prime} B^{\prime}$ sides in $k$ scale

$$
\begin{aligned}
& \frac{|A B|}{\left|A^{\prime} B^{\prime}\right|}=k, \\
& |A B|=k *\left|A^{\prime} B^{\prime}\right| .
\end{aligned}
$$

Question 17.16 Draw a triangle $\triangle A B C$ with the side

$$
|A B|=c=6 \mathrm{~cm}
$$

and with adjacent angles $\angle \alpha=30^{\circ}, \angle \beta=60^{\circ}$ to side $A B$.
Draw a second triangle $\Delta A^{\prime} B^{\prime} C^{\prime}$ with sides twice longer of the sides of triangle $\triangle A B C$.

### 17.8.1 Thales theorem

If the arms of the angle are cut by two parallel lines, the lengths of the sections delineated by the lines on one arm of the angle are proportional to the length of the sections delineated by the lines at the other arm of the angle
If the line $L_{1}$ is parallel to the line $L_{2}$, we write straight $\left\|_{1}\right\|$ straight $_{2}$ then the proportions are satisfied

$$
\begin{aligned}
& \frac{a}{b}=\frac{c}{d}, \quad \frac{a}{c}=\frac{b}{d} \\
& \frac{a}{a+b}=\frac{c}{c+d}=\frac{x}{y}
\end{aligned}
$$



Example 17.9 Calculate the height of the tree from a distance of 50m. using Thales' theorem.
Solution. Let us draw plot of tree location

where geodesian stick $x=2$ and $y$ is nunknown hight of tree
We calculate the height $y$ of tree from proportions

$$
\frac{a}{a+b}=\frac{x}{y}, \quad \text { hence } \quad y=\frac{(a+b) * x}{a}
$$

From the plot above, we find:

$$
a+b=50 \mathrm{~m} .
$$

Next we measure distance $a=10 m$, and hight of the geodesian stick $x=2 m$ and we substiute these results of mesurements to the proportion

$$
\frac{10}{50}=\frac{2}{y}, \quad \text { hence } \quad y=\frac{50 * 2}{10}=10
$$

Answer: Hight of tree equals 10 m .

### 17.8.2 Division of a segment in a given proportion

Thales' theorem is applied indivision of a segment in a given proportion.
In order to clarify division of a segment in given proportion $k$, we consider the following example.
Example 17.2 Divide segment $A B$ in the propotion 2:3


Solution. On the $A C$ arm we mark three points $D, E$ and the point $C$ with any compasses width. Next, we connect the point $C$ to the point $B$ using a ruler. We draw parallel to the segment $B C$
through the points $D$ and $E$. Thus, we get the division of the segment $A B$ by the point $F$ in the ratio 2: 3, So, from Thales' theorem we have the proportions

$$
\frac{|A F|}{|A B|}=\frac{2}{3}
$$

The geometric interpretation of this proportion is given below in the figure


Question 17.17 Calculate the height of the tree from a distance of 150 m , knowing that the height of the geodesic bar is $2 m$ and its distance from the measuring point is 10 m .

Question 17.18 Divide $A B$ in the ratio 1:3


### 17.8.3 Pythagorean theorem

There are wide applications of Pythagorean theorem. Here we apply this theorem to describe regular polygons, circles, arc measure of angles, constructions of plane figures, spatial figures, straight prisms, cylinders, cones, pyramids.


Rectangular triangle $\triangle A B C$

Theorem 17.3 In a right triangle, the sum of the squares of the hypotenuses is equal to the square of the hypotenuse

$$
\begin{equation*}
a^{2}+b^{2}=c^{2} \tag{17.5}
\end{equation*}
$$

where adjacent sides to right angle are marked by letters $a$ and $b$, and the hypotenuse by the letter c.

Question 17.19 Find the side $x$ of the rectangular triangle.


Rectangular tringle $\triangle A B C$

Question 17.20 Find the side $y$ of the rectangular triangle


Rectangular triangle $\triangle A B C$

Question 17.21 Calculate sides adjacent to right angle of the isosceles rectangular triangle knowing that the hypotenuse $c=9$.


Rectangular triangle $\triangle A B C$
Question 17.22 Calculate all sides of a rectangular triangle, knowing that $a=12 \mathrm{~cm}, b$ is 4 cm longer than $a$, and $c$ is 8 cm longer than $a$.

### 17.9 Heron's formula.

Heron's formula concerns relationship between the perimeter and the area of a triangle. Let us recall perimeter of the triangle $\triangle A B C$ is equal to the sum of the lengths of its sides

$$
O b=|A B|+|A C|+|C A| \quad \text { or } \quad O b=a+b+c
$$



The area of the triangle $\triangle A B C$ is calculated using the Heron formula. ${ }^{7}$

## Heron's formula.

$$
P_{\triangle A B C}=\sqrt{p(p-a)(p-b)(p-c)}
$$

where half of the circumference $p=\frac{a+b+c}{2}$.
Proof. Consider two identical triangles $\triangle A B C$ and $\Delta B A^{\prime} C$, as in the picture


Let us note that the area of the parallelogram $A B A^{\prime} C$ is equal

$$
P_{A B A^{\prime} C}=c * h
$$

The height $h=|D C|$ we calculate using Pythagorean theorem. Namely, the following relationships follow from the Pythagorean theorem
From the right triangle $\triangle A D C$ we have the equality

$$
h^{2}=b^{2}-(c-x)^{2}
$$

Similarly from right triangle $\Delta D A^{\prime} C$

$$
h^{2}=a^{2}-x^{2}
$$

Hence, we calculate the length of the segment $x=|A D|$

$$
\begin{aligned}
& b^{2}-(c-x)^{2}=a^{2}-x^{2} \\
& b^{2}-c^{2}+2 c * x-x^{2}=a^{2}-x^{2} \\
& 2 c * x=a^{2}-b^{2}+c^{2} \\
& x=\frac{a^{2}-b^{2}+c^{2}}{2 c}
\end{aligned}
$$

[^18]Now we calculate the square of the parallel height of $A B A^{\prime} C$ using $c$ simplified multiplication formulas

$$
\begin{aligned}
h^{2}=a^{2}-x^{2} & =a^{2}-\left(\frac{a^{2}-b^{2}+c^{2}}{2 c}\right)^{2} \\
& =\frac{4 a^{2} c^{2}-\left(a^{2}-b^{2}+c^{2}\right)^{2}}{4 c^{2}} \\
& =\frac{(2 a c)^{2}-\left(a^{2}-b^{2}+c^{2}\right)^{2}}{4 c^{2}} \\
& =\frac{\left(2 a c-a^{2}+b^{2}-c^{2}\right)\left(2 a c+a^{2}-b^{2}+c^{2}\right)}{4 c^{2}} \\
& =\frac{\left(b^{2}-(a-c)^{2}\right)\left((a+c)^{2}-b^{2}\right)}{4 c^{2}} \\
& =\frac{(b-a+c)(b+a-c)((a+c-b)(a+c+b)}{4 c^{2}} \\
& =\frac{(a+b+c-2 a)(a+b+c-2 c)((a+b+c-2 b)(a+c+b)}{4 c^{2}} \\
& =\frac{2(p-a) 2(p-c) 2(p-b) 2 p}{4 c^{2}}, \quad p=\frac{a+b+c}{2} \\
& =\frac{4 p(p-a)(p-b)(p-c)}{c^{2}}, \quad p=\frac{a+b+c}{2}
\end{aligned}
$$

Finally, the area of the triangle $\triangle A B C$ with the given side lengths is equal to half the area of the parallelogram $A B A^{\prime} C$. So we have Heron's formula for the area of the triangle $\triangle A B C$

$$
\begin{aligned}
P_{\triangle A B C}=\frac{1}{2} c * h & =\frac{c}{2} \sqrt{\frac{4 p(p-a)(p-b)(p-c)}{c^{2}}} \\
& =\sqrt{p(p-a)(p-b)(p-c)}
\end{aligned}
$$

Example 17.10 Find the perimeter and area of the triangle $\triangle A B C$ with the sides

$$
a=|A B|=3 \mathrm{~cm}, \quad b=|C A|=4 \mathrm{~cm}, \quad c=|A B|=5 \mathrm{~cm}
$$

Solution. The circumference of the triangle $\triangle A B C$ is equal to the sum of the lengths of its sides

$$
\begin{aligned}
& O b=a+b+c=3 \mathrm{~cm}+4 \mathrm{~cm}+5 \mathrm{~cm}=12 \mathrm{~cm} \\
& \text { half of the circumference } \\
& p=\frac{a+b+c}{2}=\frac{3 \mathrm{~cm}+4 \mathrm{~cm}+5 \mathrm{~cm}}{2}=6 \mathrm{~cm}
\end{aligned}
$$

Area of the triangle $\triangle A B C$ we calculate from Heron's formula

$$
\begin{aligned}
P_{A B C} & =\sqrt{p(p-a)(p-b)(p-c)} \\
& =\sqrt{6 \mathrm{~cm}(6 \mathrm{~cm}-3 \mathrm{~cm})(6 \mathrm{~cm}-4 \mathrm{~cm})(6 \mathrm{~cm}-5 \mathrm{~cm})} \\
& ==\sqrt{6 * 3 * 2 * 1 \mathrm{~cm}^{4}}=\sqrt{36 \mathrm{~cm}^{4}}=6 \mathrm{~cm}^{2} .
\end{aligned}
$$

Question 17.23 Find the perimeter and area of the triangle $\triangle A B C$ with the sides

$$
a=|A B|=6 c m, \quad b=|C A|=8 \mathrm{~cm}, \quad c=|A B|=10 \mathrm{~cm}
$$

### 17.10 Quadrilaterals

Consider a quadrilateral $A B C D$ with vertices $A, B, C, D$ with four sides $a, b, c, d$, and angles $\angle A B C$ at vertex $B$, angle $\angle B C D$ at vertex $C, \angle \angle C D A$ at vertex $D$.

The sum of the lengths of any selected three sides of the quadrilateral is not less than the length of the fourth side, we write

$$
|A B|+|B C|+|C D| \geq|A D|
$$

The sum of the angles of a quadrilateral is equal $360^{\circ}$, in the arc measure $2 \pi$, we write

$$
\angle A B C+\angle B C D+\angle C D A+\angle D A B=360^{\circ}
$$



Question 17.24 Measure the sides and angles of this quadrilateral. Calculate the perimeter and sum of the angles of the quadrilateral quadrilateral $A B C D$.

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### 17.10.1 Regular quadrilateral. Square.

The square $A B C D$ is a regular quadrilateral with four sides a and four right angles $90^{\circ}$ or in the arc measure $\frac{\pi}{2}$.


The square has two diagonals $A C$ and $B D$, which intersect at a right angle equal to $90^{\circ}$ or in an arc measure $\frac{\pi}{2}$. We calculate the length of the diagonal from the Pythagorean formation

$$
|A C|^{2}=|B C|^{2}=a^{2}+a^{2}=2 a^{2}, \quad|A C|=|B C|=a \sqrt{2}
$$

The radius of the circle inscribed in the square is equal to the half of the side

$$
r=\frac{a}{2}
$$

[^19]The radius of the circle described on the square is equal to half the diagonal

$$
R=\frac{a \sqrt{2}}{2}
$$

The area of the square

$$
P_{A B C D}=a * a^{2}, \quad \text { circuit square } O b=4 * a
$$

Question 17.25 Calculate the perimeter $O b$ and the area $P$ of the square, the length of the diagonals, the radius $r$ of the circle inscribed in the square and the radius $R$ of the circle inscribed in the square, if the side of the square is $a=4$..

### 17.10.2 Rectangle.

The rectangle $A B C D$ has four pairs of equal sides

$$
a=c, b=d
$$

and four right angles equal to $90^{\circ}$
We calculate the diagonal of the rectangle from the Pythagorean theorem

$$
|A C|=|B D|=\sqrt{a^{2}+b^{2}}
$$

Rectangle area

$$
P_{A B C D}=a * b .
$$

The perimeter of the rectangle $\mathrm{Ob}=2 * a+2 * b$


The circumscribed circle of the rectangle has a radius $R$ equal to half the diagonals

$$
R=\frac{1}{2}|A C|=\frac{1}{2}|B C|=\frac{1}{2}\left(a^{2}+b^{2}\right)
$$

On the other hand, there is no circle inscribed in a rectangle, except for a square which is a special rectangle with equal sides.

### 17.10.3 Paralleogram.

Parallelogram $A B C D$ has two pairs of equal sides

$$
a=c, \quad b=d
$$

and two pairs equal angles

$$
\alpha=\angle D A B=\angle B C D, \quad 180^{\circ}-\alpha=\angle A B C=\angle C D A
$$



The height of the parallelogram is denoted by the letter $h$.
The area of the parallelogram

$$
P_{A B C D}=a * h .
$$

Indeed, note that the area of the parallelogram $A B C D$ is equal to the area of the rectangle $E \hat{B} C D$. It means that

$$
P_{A \hat{B} C D}=P_{A B C D}=a * h .
$$

and perimeter of the parallelogram

$$
O b=2 * a+2 * b .
$$

### 17.10.4 Rhombus.

Rhombus $A B C D$ has four equal sides

$$
a=|A B|=B C|=|C D|
$$

and four pairs of angles equal

$$
\alpha=\gamma, \quad \beta=\delta .
$$

The height of the rhombus is denoted by the letter $h$.
Note that the circumference of the rhombus

$$
O b=4 * a .
$$

Rhombus field

$$
P=a * h .
$$



### 17.10.5 Trapezoid

Trapezoid $A B C D$

is a quadrilateral with the length of the lower base $a=|A B|$ parallel to the upper base with the length $b=|C D|$.
The field of the trapezoid

$$
P_{A B C D}=\frac{1}{2}(a+b) * h .
$$

Indeed, notice that the area of the trapezoid $P_{A B C D}$ is equal to the sum of the area of the parallelogram

$$
\hat{P}_{A E C D}=(b * h
$$

and triangle fields

$$
P_{\hat{A} B C}=\frac{1}{2}(a-b) * h .
$$

So the trapezoid fieldu

$$
\begin{aligned}
P_{A B C D} & =\hat{P}_{A E C D}+P_{\hat{A} B C} \\
& =b * h+\frac{1}{2}(a-b) * h \\
& =\frac{1}{2}(a+b) * h
\end{aligned}
$$

Trapezoid circumference

$$
O b=|A B|+|B C|+|C D|+|A C| .
$$

10

### 17.10.6 Deltoid.

Deltoid is a quadrilateral with pairs of equal sides

$$
|A B|=|A D|, \quad|C D|=|A D|
$$

$i$ with angles $\angle A D C=\triangle A B C$
Deltoid has two perpendicular diagonals $L_{1}$ and $L_{2}$, one of them is the symmetry of the other, as

[^20]below in the picture


The deltoid area is half the product of the diagonals

$$
P_{\text {Deltoid }}=\frac{1}{2} L_{1} * L_{2}
$$

Indeed, we note that the field of the deltoid $P_{A B C D}$ is equal to the sum of the fields of the triangles $\triangle A B D$ and $\triangle B C D$

$$
\begin{aligned}
P_{\text {Deltoid }}=P_{A B D}+P_{D B C} & =\frac{1}{2} L_{2} *|A O|+\frac{1}{2} L_{2} *|O C| \\
& =\frac{1}{2} \underbrace{(|A O|+|O C|)}_{L_{1}} \\
& =\frac{1}{2} * L_{1} * L_{2} .
\end{aligned}
$$

11

### 17.10.7 A circle described on a quadrilateral.

The conditions for the existence of the circle described on the quadrilateral and the circle inscribed in the quadrilateral are given below.
Namely, let us consider the quadrilateral $A B C D$ inscribed with a circle of radius $R$ and center at the point $O$.


As we know, the center angles based on the same arc are equal.

[^21]Thus, we note in the figure that

$$
\begin{array}{lllll}
\zeta=\psi & \text { based } & \text { on } & \text { arc } & \widehat{A B} \\
\eta=\gamma & \text { based } & \text { on } & \text { arc } & \widehat{A D} \\
\delta=\alpha & \text { based } & \text { on } & \text { arc } & \widehat{D C}  \tag{17.6}\\
\beta=\phi & \text { based } & \text { on } & \text { arc } & \widehat{B C}
\end{array}
$$

The necessary and sufficient condition of ploting a circle on a given quadrilateral with vertices $A, B, C, D$ is given in the form of the following theorem

Theorem $17.4 A$ circle can be described on the quadrilateral $A B C D$ vertices $A, B, C, D$ if and only if the sums of the opposite lyingg angles are equal to

$$
\begin{equation*}
\angle A B C+\angle C D A=\angle B C A+\angle D A B \tag{17.7}
\end{equation*}
$$

${ }^{12}$ Proof. Proof from the theorem derives from the equality (17.6) of the angles $\alpha, \beta, \gamma, \delta, \eta, \zeta$ which form diagonals with the sides of the quadrilateral. Now we check for equality (17.7)

$$
\begin{aligned}
\angle A B C+\angle C D A & =\underbrace{(\delta+\gamma)}_{\angle A B C}+\underbrace{(\phi+\psi)}_{\angle C D A} \\
& =(\alpha+\eta)+(\beta+\zeta) \\
& =\underbrace{(\alpha+\beta)}_{\angle D A C}+\underbrace{(\eta+\zeta)}_{B C D} \\
& =\angle B C A+\angle D A B
\end{aligned}
$$

The formula for the area of a quadrilateral inscribed in a circle

$$
P_{A B C D}=\sqrt{(p-a)(p-b)(p-c)(p-d)}
$$

where the lengths of the sides

$$
a=|A B|, \quad b=|B C|, \quad c=|C D|, \quad d=|D A|,
$$

and the letter $p$ stands for half the circumference of the quadrilateral

$$
p=\frac{1}{2}(a+b+c+d)=\frac{1}{2}(|A B|+|B C|+|C D+|A D|) .
$$

### 17.10.8 Circle inscribed in a quadrilateral

The description of the circle inscribed in the quadrilateral with the vertices $A, B, C, D$ let's start with the following observations:
(a) The sides of the quadrilateral are tangent to the circle inscribed at the points of tangency $A^{*}, B^{*}, C^{*}, D^{*}$
(b) Tangents to the circle drawn from the vertices of the quadrilateral mark the segments in pairs of equal length, we write

$$
\begin{array}{ll}
x=\left|A, A^{*}\right|=\left|A D^{*}\right|, & y=\left|A^{*} B\right|=\left|B C^{*}\right| \\
z=\left|B^{*} C\right|=\left|C C^{*}\right|, & t=\left|C^{*} D\right|=\left|D D^{*}\right| \tag{17.8}
\end{array}
$$

[^22]

The necessary and sufficient condition for a circle to inscribe in a given quadrilateral with the vertices $A, B, C, D$ we write in the form of the following theorem
Theorem 17.5 $A$ circle can be entered in the quadrilateral $A B C D$ with vertices $A, B, C, D$ if and only if the sums of the lengths of the opposite sides are equal to

$$
\begin{equation*}
|A B|+|C D|=A D|+|B C| \tag{17.9}
\end{equation*}
$$

Proof. The proof of the theorem follows from the properties (a) and (b) to the tangent to the circle and from the equality (17.8), namely, we check that the left side of the equality(17.9)

$$
\begin{aligned}
|A B|+|C D| & =\underbrace{(x+y)}_{\text {bok }|A B|}+\underbrace{(z+t)}_{\text {bok }|C D|} \\
& =\underbrace{(x+t)}_{\text {bok }|A D|}+\underbrace{(y+z)}_{\text {bok } B C \mid} \\
& =|A D|+|B C|,
\end{aligned}
$$

is equal to the right side of the equality (17.9). ${ }^{13}$
Area of the quadrilateral and the radius of the circle inscribed in the quadrilateral. Note that the radius $r$ of the circle inscribed in the quadrilateral is equal to the heights triangles

$$
\triangle A O B, \quad \triangle B C O, \quad \triangle D C O, \quad \triangle D A O
$$

sideways

$$
[A, B], \quad[B, C], \quad[C, D], \quad ;[D, A]
$$

quadrilateral $A B C D$.
So the areas of these triangles are equal respectively

$$
\begin{aligned}
& P_{\triangle A B O}=\frac{1}{2} r *|A B| \\
& P_{\triangle B C O}=\frac{1}{2} r *|B C| \\
& P_{\triangle C D O}=\frac{1}{2} r *|C D| \\
& P_{\triangle D A O}=\frac{1}{2} r *|D A|
\end{aligned}
$$

The area of the quadrilateral $A B C D$ is equal to the sum of the areas of the four triangles, we write

$$
\begin{aligned}
P_{A B C D} & =P_{\triangle A B O}+P_{\triangle B C O}+P_{\triangle C D O}+P_{\triangle D A O} \\
& =\frac{1}{2} r *|A B|+\frac{1}{2} r *|B C|+\frac{1}{2} r *|C D|+\frac{1}{2} r *|D A| \\
& =\frac{1}{2} r(|A B|+|B C|+|C D|+|D A|) \\
& =r * p
\end{aligned}
$$

[^23]where the letter $p$ denotes half the circumference of the quadrilateral $A B C D$.
$$
p=\frac{1}{2}(|A B|+|B C|+|C D|+|D A|) .
$$

### 17.11 Vector product and its applicatios

In this section we will apply vector product to compute the area of an arbitrary quadrilateral. $A B C D$ with given vertices $A, B, C, D$ in Cartesian coordinates.

Let us clarify that Vector product is described in the next chapter on spatial geometry. In this chapter, concerning plane geometry, we use the vector product to calculate the area of an arbitrary quadrilateral. Let us consider two vectors

$$
\vec{v}=\left(v_{1}, v_{2}, v_{3}\right), \quad \text { and } \quad \vec{w}=\left(w_{1}, w_{2}, w_{3}\right)
$$

in three-dimensional space


The result of multiplying the vector $\vec{v}$ by the vector $\vec{w}$ is the third vector $\vec{v} \times \vec{w}$, the coordinates of which are calculated from expansion of the matrix from the coordinates of the vectors

$$
\left\{\begin{array}{lll}
1 & 1 & 1 \\
v_{1} & v_{2} & v_{3} \\
w_{1} & w_{2} & w_{3}
\end{array}\right\}
$$

Namely the product

$$
\vec{v} \times \vec{w}=\left[\operatorname{Det}\left\{\begin{array}{ll}
v_{2} & v_{3} \\
w_{2} & w_{3}
\end{array}\right\},-\operatorname{Det}\left(\left\{\begin{array}{ll}
v_{1} & v_{3} \\
w_{1} & w_{3}
\end{array}\right\}\right), \operatorname{Det}\left\{\begin{array}{ll}
v_{1} & v_{2} \\
w_{1} & w_{2}
\end{array}\right\}\right]
$$

where determinants are defined by formulas

$$
\begin{aligned}
& \operatorname{Det}\left\{\begin{array}{ll}
v_{2} & v_{3} \\
w_{2} & w_{3}
\end{array}\right\}=v_{2} * w_{3}-v_{3} * w_{2}, \\
& -\operatorname{Det}\left\{\begin{array}{ll}
v_{1} & v_{3} \\
w_{1} & w_{3}
\end{array}\right\}=-\left(v_{1} * w_{3}-v_{3} * w_{1}\right), \\
& \operatorname{Det}\left\{\begin{array}{ll}
v_{1} & v_{2} \\
w_{1} & w_{2}
\end{array}\right\}=v_{1} * w_{2}-v_{2} * w_{1}
\end{aligned}
$$

Hence, we get the formula for the coordinates of the vector product

$$
\begin{equation*}
\vec{v} \times \vec{w}=\left[v_{2} * w_{3}-v_{3} * w_{2},-\left(v_{1} * w_{3}-v_{3} * w_{1}\right), v_{1} * w_{2}-v_{2} * w_{1}\right] . \tag{17.10}
\end{equation*}
$$

Vector $\vec{v} \times \vec{w}$ is perpendicular to the vectors $\vec{v} i \vec{w}$, we write

$$
\vec{v} \times \vec{v} \perp \vec{w}, \quad \vec{w} \times \vec{v} \perp \vec{w}
$$

We know that vectors are perpendicular if and only if their scalar product is equal to zero

$$
(\vec{v}, \vec{w})=0
$$

Let us calculate the scalar product

$$
\begin{aligned}
(\vec{v}, \vec{v} \times \vec{w}) & =\left(\left[v_{1}, v_{2}, v_{3}\right],\left[v_{2} * w_{3}-v_{3} * w_{2},-\left(v_{1} * w_{3}-v_{3} * w_{1}\right), v_{1} * w_{2}-v_{2} * w_{1}\right]\right) \\
& =v_{1}\left(v_{2} * w_{3}-v_{3} * w_{2}\right)-v_{2}\left(v_{1} * w_{3}-v_{3} * w_{1}\right)+v_{3}\left(v_{1} * w_{2}-v_{2} * w_{1}\right) \\
& =\left(v_{1} v_{2} w_{3}+v_{2} v_{3} w_{1}+v_{3} v_{1} w_{2}\right)-\left(v_{1} v_{3} w_{2}+v_{2} v_{1} w_{3}+v_{3} v_{2} w_{1}\right)=0
\end{aligned}
$$

Indeed, the scalar product

$$
(\vec{v}, \vec{v} \times \vec{w})=0
$$

equals to zero
The length of the vector $\vec{v} \times \vec{w}$ is equal to the area of the parallelogram with sides $\vec{v}$ and $\vec{w}$.
${ }^{14}$ So the length of the vector

$$
|\vec{v} \times \vec{w}|=\sqrt{\left|v_{2} * w_{3}-v_{3} * w_{2}\right|^{2}+\left|-\left(v_{1} * w_{3}-v_{3} * w_{1}\right)\right|^{2}+\left|v_{1} * w_{2}-v_{2} * w_{1}\right|^{2}}
$$

### 17.12 Examples of application of vector product.

As one of possible applications of vector product we calculate area of a quadrilateral. Namely let us consider a quadrilateral $A B C D$

spaned on vectors

$$
\begin{array}{ll}
\vec{v}=\left[v_{1}, v_{2}, v_{3}\right]=\overrightarrow{A B}, & \vec{w}=\left[w_{1}, w_{2}, w_{3}\right]=\overrightarrow{A D} \\
\vec{Q}=\left[z_{1}, z_{2}, z_{3}\right]=\overrightarrow{C B}, & \vec{t}=\left[t_{1}, t_{2}, t_{3}\right]=\overrightarrow{C D}
\end{array}
$$

where the coordinates of the vectors $\vec{v}, \vec{w}, \vec{Q}, \vec{t}$ are determined by the differences in the coordinates of the vertices $A, B, C, D$ of the quadrilateral $A B C D$

$$
\begin{array}{lll}
v_{1}=b_{1}-a_{1}, & v_{2}=b_{2}-a_{2}, & v_{3}=b_{3}-a_{3}, \\
w_{1}=d_{1}-a_{1}, & w_{2}=d_{2}-a_{2}, & w_{3}=d_{3}-a_{3} \\
z_{1}=b_{1}-c_{1}, & z_{2}=b_{2}-c_{2}, & z_{3}=b_{3}-c_{3} \\
t_{1}=d_{1}-c_{1}, & t_{2}=d_{2}-c_{2}, & t_{3}=d_{3}-c_{3} .
\end{array}
$$

Using the vector product (cf. (17.10)) we can calculate the area of any quadrilateral with the given coordinates of its vertices. Namely, the area of a convex quadrilateral $A B C D$ is equal to half of the sum of the vector product of vectors ${ }^{15}$

$$
\begin{equation*}
P_{A B C D}=\frac{1}{2} \vec{v} \times \vec{w}+\frac{1}{2} \vec{Q} \times \vec{t} \tag{17.11}
\end{equation*}
$$

[^24]Example 17.11 Find the area of the quadrilateral $A B C D$ determined by vectors

and the product of vectors

$$
\begin{aligned}
\vec{Q} \times \vec{t} & =[-6 * 0-3 * 0,-(0 * 0-3 * 0), 0 * 3-6 * 3] \\
& =[0.0,-18]
\end{aligned}
$$

We calculate the area of the quadrilateral $A B C D$ as the sum of half the vector product of the vectors $\vec{v}, \vec{w}$ and $\vec{Q}, \vec{t}$.

$$
\begin{aligned}
P_{A B C D} & =\frac{1}{2}|\vec{v} \times \vec{w}|+\frac{1}{2}|\vec{Q} \times \vec{t}| \\
& \left.=\frac{1}{2} \sqrt{0^{2}+0^{2}+9^{2}}+\frac{1}{2} \right\rvert\, \sqrt{0^{2}+0^{2}+(-18)^{2}} \\
& =\frac{1}{2} * 9+\frac{1}{2} * 18 \\
& =4.5+9=13.5
\end{aligned}
$$

16

[^25]Example 17.12 Calculate the area of the quadrilateral $A B C D$ spanning by the vectors

$$
\begin{aligned}
& \vec{v}=[4.0 .0,0]=\overrightarrow{A B}, \\
& \vec{w}=[-0.5,2.4,0]=\overrightarrow{A D}, \\
& \vec{Q}=[-1.0,4.2,0]=\overrightarrow{C B}, \\
& \vec{t}=[-3.5,-1.8,0]=\overrightarrow{C D}
\end{aligned}
$$



We calculate the vector products of $\vec{v} \times \vec{w}$ and $\vec{Q} \times \vec{t}$ using formulas (cf. (17.10))

$$
\begin{aligned}
\vec{v} \times \vec{w} & =[0 * 0-0 * 2.4,-(4 * 0+0.5 * 0), 4 * 2.4+0.5 * 0] \\
& =[0.0,9.6]
\end{aligned}
$$

and the product of vectors

$$
\begin{aligned}
\vec{Q} \times \vec{t} & =[4.2 * 0+1.8 * 0,-((-1) * 0-(-3.5) * 0), 1 * 1.8+4.2 * 3.5] \\
& =[0.0,16.5]
\end{aligned}
$$

We calculate the area of the quadrilateral $A B C D$ as the sum of half the vector product of the vectors $\vec{v}, \vec{w}$ and $\vec{Q}, \vec{t}$.
Namely

$$
\begin{aligned}
P_{A B C D} & =\frac{1}{2}|\vec{v} \times \vec{w}|+\frac{1}{2}|\vec{Q} \times \vec{t}| \\
& =\frac{1}{2} \sqrt{0^{2}+0^{2}+9.6^{2}}+\frac{1}{2} \sqrt{0^{2}+0^{2}+16.5^{2}} \\
& =\frac{1}{2} * 9.6+\frac{1}{2} * 16.5 \\
& =4.8+8.25=13.05
\end{aligned}
$$

Question 17.26 Calculate the length of the vectors

$$
\text { (i) } \vec{v}=[3,0,4], \quad \vec{w}=[8,-6,0]
$$

(ii) Calculate the vector product $\vec{v} \times \vec{w}$ of vectors

$$
\vec{v}=[3,0,4], \quad \vec{w}=[8,-6,0]
$$

(iii) Check if the vectors $\vec{v}$ and $\vec{w} \times \vec{w}$ are perpendicular by using scalar product

$$
\vec{v} \perp \vec{w} \times \vec{w}
$$

Question 17.27 Check if the vector

$$
\vec{w}=\left[w_{1}, w_{2}, 0\right]
$$

is perpendicular to the vector

$$
\vec{v} \times \vec{w},
$$

where

$$
\vec{v}=\left[v_{1}, v_{2}, 0\right]
$$

### 17.13 Regular plane figures

Regular figures on a plane are plane figures with all sides and all angles equal.

### 17.13.1 Regular triangle

An equilateral triangle is a regular triangle Equilateral triangle has all sides equal and all angles $\alpha=60^{\circ}$, in arc measure $\alpha=\frac{\pi}{3}$ as in the picture


Regular triangle $\triangle A B C$
The height h triangle $\triangle A B C$ is the bisector of angle $\alpha$ and halves the base a at point $D$. Similarly, the heights of an equilateral triangle, dropped to the other sides, divide the bases into half and intersect at the point $O$, that is in the center of the circle inscribed in the equilateral triangle. The intersection of $O$ divides these amounts by 1:3. That is, the following proportion holds

$$
\frac{|D O|}{|D C|}=\frac{1}{3}, \quad|D C|=h
$$

Hence we have

$$
|D O|=\frac{1}{3} h, \quad \text { i } \quad|O C|=\frac{2}{3} h
$$

We calculate the height of $h$ triangle by the Pythagorean theorem $\triangle A B C$

$$
h^{2}=a^{2}-\left(\frac{a}{2}\right)^{2}=\frac{3}{4} a^{2},
$$

Thus height of the equilateral triangle

$$
h=\frac{a \sqrt{3}}{2}
$$

We derive formula for the area of an equilateral triangle

$$
\begin{aligned}
& P=h * \frac{a}{2}=\frac{a \sqrt{3}}{2} * \frac{a}{2}=\frac{a^{2} \sqrt{3}}{4} \\
& P=\frac{a^{2} \sqrt{3}}{4}
\end{aligned}
$$

Thus area of the equilateral triangle with side $a$ is given by formula

$$
P=\frac{a^{2} \sqrt{3}}{4}
$$

### 17.13.2 Square.

The square $A B C D$ is with four sides equal a and four right angles $90^{\circ}$ or $\frac{\pi}{2}$. Below, the circle inscrobed in the square of radius $r=\frac{a}{2}$


The square has two diagonals $A C$ and $B D$, which intersect at a right angle equal to $90^{\circ}$ or in an arc measure $\frac{\pi}{2}$. We calculate the length of the diagonal from the Pythagorean formula

$$
|A C|^{2}=|B C|^{2}=a^{2}+a^{2}=2 a^{2}, \quad|A C|=|B C|=a \sqrt{2}
$$

Radius $r$ of the circle inscribed in the square is equal to half of the side

$$
r=\frac{a}{2}
$$

Radius $R$ of the circle described on the square is equal to half the diagonal

$$
R=\frac{a \sqrt{2}}{2}
$$

Area of square

$$
S=a * a^{2}, \quad \text { perimeter thesquare } O b=4 * a .
$$

On the graph beloww, thr circle described on the square wih radius $R=\frac{|A C|}{2}$ half of diagonal $[A, C]$


Question 17.28 Calculate the perimeter, length of the diagonals and the area of the square of side $a=4$

### 17.13.3 Regular pentagon

Regular pentagon has sides equal $a$ and equal angles $\angle E A B=\alpha=108^{0}$ or $\alpha=\frac{3 \pi}{5}$.


$$
\angle E A B=\angle A B C=\angle B C D=\angle C D E=\angle D E A=\alpha=108^{0}
$$

The area of a regular pentagon. The area of a regular pentagon consists of 5 areas of isosceles and congruent triangles of height $h$ and the base of $a$.
The area of one of the five triangles $\triangle A O E$

$$
P_{\triangle A O E}=\frac{1}{2} a * h
$$

where hight

$$
\begin{aligned}
h=\frac{1}{2} a * \operatorname{ctg} 36^{0} & =\frac{a}{2} * \frac{1}{\sqrt{5-2 \sqrt{5}}} \\
& =\frac{a}{2} * \frac{\sqrt{5+2 \sqrt{5}}}{\sqrt{25-20}} \\
& =\frac{a}{2} * \frac{\sqrt{5+2 \sqrt{5}}}{\sqrt{5}} \\
& =\frac{a}{2} * \frac{\sqrt{25+10 \sqrt{5}}}{5}
\end{aligned}
$$

Thus, the area of a regular pentagon with the side $a$ is determined by the formula

$$
P=5 * P_{\triangle A O E}=5 * \frac{1}{2} * a * h=\underbrace{\frac{a^{2}}{4} \sqrt{25+10 * \sqrt{5}}}_{\text {Regularfield fivetimes }}
$$

Radius of the circle described on a regular pentagon. Radius $R=|A O|$ of the circle described on the regular pentagon, we calculate from the right triangle $\triangle A O F$ using the Pythagorean theorem.

Namely, the square $R^{2}$ is equal to the sum of the squares $|F O|^{2}+|F A|^{2} \mid$, we write

$$
\begin{aligned}
R^{2} & =\underbrace{|F O|^{2}}_{h^{2}}+|F A|^{2} \mid \\
& =h^{2^{2}}+\left(\frac{a}{2}\right)^{2} \\
& =\underbrace{\left(\frac{a}{10} * \sqrt{25+10 \sqrt{5}}\right)^{2}}_{h^{2}}+\left(\frac{a}{2}\right)^{2} \\
& =\frac{a^{2}}{100}(25+10 \sqrt{5})+\frac{25 a^{2}}{100} \\
& =\frac{a^{2}}{100}(50+10 \sqrt{5})
\end{aligned}
$$

Hence, the radius of the descibed circle on a regular pentagon is given by the formula

$$
\begin{aligned}
R & =\sqrt{\frac{a^{2}}{100}(50+10 \sqrt{5})} \\
& =\underbrace{\frac{a}{10} \sqrt{50+10 \sqrt{5}}}_{R}
\end{aligned}
$$

Radius of the circle inscribed in a regular pentagon. Radius $r=|F O|$ of the circle inscribed in a regular pentagon, we will calculate from the right triangle $\triangle A O F$ using Pythagorean theorem. Namely, the square $r^{2}$ is equal to the sum of the squares $|F O|^{2}+|F A|^{2} \mid$, we write Note that the radius $r=|F O|=h$ equals

$$
r=\frac{a}{10} * \sqrt{25+10 \sqrt{5}}
$$

The diagonals of a regular pentagon. A regular pentagon has 5 equal diagonals of length

$$
\begin{aligned}
d=|E C| & =2 a * \cos \frac{\pi}{5} \\
& =2 a * \frac{1+\sqrt{5}}{4} \\
& =\frac{a}{2}(1+\sqrt{5})
\end{aligned}
$$

### 17.13.4 Regular hexagon

Regular hexagon has six sides equal to a and six equal angles $\alpha=120^{\circ}$ or $\frac{2 \pi}{3}$

$$
\text { andsixsides }|A B|=|B C|=|C D|=|D E|=|E F|=|F E|=a=R
$$

where $R$ is the radius of the circumscribed circle on the regular hexagon with vertices $A, B, C, D, E, F$ and angles

$$
\angle A B C=\angle B C D=\angle C D E=\angle D E F=\angle E F C=\angle F A B=\alpha=120^{\circ} . \text { or } \alpha=\frac{2 \pi}{3}
$$



## Regular hexogen

Six sides of length a each
Six angles $\alpha=120^{\circ}$ each
Six congruent regular triangles with common vertices at point $O$

The construction of a regular hexagon with the help of a compasses and a ruler is very simple, Namely, let the side of the hexagon be given as a segment $[A, B]$
Place a compasses at any chosen point $O$, and with the compasses's width equal to the segment $[A, B]$ we draw a circle with the radius $R=|A B|=a$ equal to the side of the hexagon $A B C D E F$. Next, we place a compasses at a point in the $A$ i circle with the compasses width $R=a$ we draw an arc that intersects the circle at $B$, then we place the compasses at $B$ and we draw an arc that intersects the circle at $C$, then we place the compasses at $C$ and we draw an arc intersecting the circle at at $D$, we place the compasses at $D$ and draw an arc intersecting the circle at $E$, then we place the compasses at $E$ and draw an arc intersecting the circle at $F$.

We join the points $A, B, C, D, E, F$ on a circle with a ruler. Thus we drew a regular hexagon $A B C D E F$.

Note that a regular hexagon consists of 6 equilateral triangles with sides equal to $R$ and all angles $60^{\circ}$ or $\frac{\pi}{3}$.
Let us list all 6 of congruent triangles

$$
\triangle A B O, \triangle B C O, \triangle C D O, \triangle D E O, \triangle E F O
$$

whose heights are equal to $h=r=|O G|$ to the radius of the circle inscribed in the hexagon. The height $h$ is calculated by applying the Pythagorean theorem to a right triangle $\triangle A G O$. Namely

$$
h^{2}=R^{2}-\left(\frac{R}{2}\right)^{2}=\frac{3 R^{2}}{4}, \quad h=\frac{R \sqrt{3}}{2} .
$$

The area of a regular hexagon consists of 6 areas of equilateral triangles with different sides $a=$ $R=|A O|$.
The area of one equilateral triangle is equal to

$$
P_{\Delta}=\frac{1}{2} a * h=\frac{a^{2}}{4} \sqrt{3}
$$

So the area of the hexagon is equal to

$$
P=6 * P_{\Delta}=6 * \frac{a^{2}}{4} \sqrt{3}=\underbrace{\frac{3 a^{2}}{2} \sqrt{3}}_{\text {area P hexagon }} .
$$

The perimeter of a regular hexagon is equal to

$$
O b=6 * a \quad \text { lub } \quad O b=6 * R, \text { bo } \quad a=R .
$$

### 17.13.5 Regular octagon

Regular octagon about eight equal sides

$$
|A B|=|B C|=|C D|=|D E|=|E F|=|F E|=|F G|=a
$$

and with eight equal angels

$$
\begin{aligned}
\angle A B C=\angle B C D=\angle C D E & =\angle D E F=\angle E F G=\angle F G H \\
& =\angle G H A=\angle H A B=\alpha=135^{\circ}
\end{aligned}
$$

in the arc measure $\alpha=\frac{3 \pi}{4}$.

### 17.13.6 Construction of a regular octagon.

We will make the construction of an octagonal shape with a given side using a compasses and a ruler.

1. Construct a square with side $a=|A B|$ using a compasses and a ruler. ${ }^{17}$

2. We draw diagonals and symmetricals of the sides of the square. The diagonal and symmetrical sides of the square intersect at one point $O$.

3. On the lengthening of the symmetrical base of the square we place the compasses width equal to half the diagonal, that is the segment $|A O|$, placing the compasses at the point $O$ of the intersection of the diagonals. We denote $E$ octagon apex as on the grath below

[^26]
4. With the compasses width equal to the radius $R=|P E|$ we draw a circle, placing the compasses at the point $O$. Then we extend the diagonals of the square and the midpoints of the sides to the points of intersection $A^{*}, B^{*}, C, D, E, F, G, H$ with a circle, that is to the vertices of an octagon of a formal $A^{*} B^{*} C D E F H$ with a given side $a=|A B|$.
We connect the tops $A^{*}, B^{*}, C, D, E, F, G, H$ eight with a ruler. In this way, we constructed a regular octagon

## $A^{*} B^{*} C D E F G H$

with the radius of the circumscribed circle $R=|P E|$.

with the radius of the circumscribed circle ${ }^{A^{*}}$

$$
\angle O E P=22.5^{0} \sim \frac{\pi}{8}, \quad \angle E P O=67.5^{0} \sim \frac{3 \pi}{8} .
$$

Area of a regular octagon. Every regular octagon is composed of 8 congruent and isosceles triangles with arms equal to the radius $R$ of the circle circumscribed octagon and all angles $135^{\circ} \sim \frac{3 \pi}{4}$.

The 8 regular triangles
$\Delta A^{*} B^{*} O, \Delta B^{*} C O, \triangle C D O, \triangle D E O$,
$\Delta E F O, \Delta F G O, \Delta G H O, \Delta H A^{*} O$
whose heights are equal to $h=r=|O Q|$ to the radius $r=h$ of the circle inscribed in the octagon. Applying the Pythagorean theorems to a right triangle $\triangle O Q E$ we calculate the square of the height

$$
\begin{aligned}
h^{2}=|O E|^{2}-|Q E|^{2} & =\underbrace{\left(\frac{a}{2}+\frac{a}{2} \sqrt{2}\right)^{2}+\left(\frac{a}{2}\right)^{2}}_{R^{2}}-\left(\frac{a}{2}\right)^{2} \\
& =\frac{a^{2}}{4}(1+\sqrt{2})^{2}
\end{aligned}
$$

Hence, we calculate the heights and the radius of the circle inscribed in the octagon

$$
h=r=\frac{a}{2}(1+\sqrt{2})
$$

Radius $R$ of a circle described on an octagon. The radius of a circle described on an octagon results from the construction of a regular octagon. Its value is calculated by applying the Pythagorean theorem to a right triangle $\triangle O Q E$. Namely, the square of the hypotenuse $R=|E O|$ is equal to

$$
\begin{aligned}
R^{2}=|E O|^{2} & =|Q O|^{2}+|E S|^{2}=h^{2}+\left(\frac{a}{2}\right)^{2} \\
& =\left(\frac{a}{2}(1+\sqrt{2})\right)^{2}+\left(\frac{a}{2}\right)^{2} \\
& =\frac{a^{2}}{4}(3+2 \sqrt{2})+\frac{a^{2}}{4}=a^{2}+\frac{a^{2}}{2} \sqrt{2} \\
& =a^{2}\left(1+\frac{1}{2} \sqrt{2}\right)
\end{aligned}
$$

Where do we calculate the radius of the circle described on the regular octagon

$$
R=a \sqrt{\frac{2+\sqrt{2}}{2}}
$$

The area of a regular octagon. The area of a regular octagon consists of 8 areas of equilateral triangles with the length base $a$ and sides of length $R$.
The area of one isosceles triangle is equal to

$$
P_{\Delta}=\frac{1}{2} a * h=\frac{a^{2}}{4}(1+\sqrt{2})
$$

So the area of an octagon is equal to

$$
P=8 * P_{\Delta}=8 * \frac{a^{2}}{4}(1+\sqrt{2})=\underbrace{2 a^{2}(1+\sqrt{2})}_{\text {area } P \text { octagon }} .
$$

The perimeter of a regular octagon is equal to $O b=8 * a$.
The radius $r$ of a circle inscribed in an octagon and a radius of $R$ of a circle inscribed on a regular octagon with a given side, we will calculate using trigonometric relationships in a right triangle $\triangle O Q E$. Namely, the radius of the circle inscribed in the octagon

$$
r=\frac{1}{2} a * \operatorname{ctg} \frac{\pi}{8}
$$

The value of the ctg $\frac{\pi}{8}$ function is calculated using the trigonometric identity

$$
\operatorname{ctg} \alpha-\frac{1}{\operatorname{ctg} \alpha}=\operatorname{ctg} 2 \alpha
$$

for $\alpha=\frac{\pi}{8}$

$$
\begin{aligned}
\operatorname{ctg} \frac{\pi}{8}-\operatorname{tg} \frac{\pi}{8} & =\frac{\cos \frac{\pi}{8}}{\sin \frac{\pi}{8}}-\frac{\sin \frac{\pi}{8}}{\cos \frac{\pi}{8}} \\
& =\frac{\cos ^{2} \frac{\pi}{8}-\sin \frac{\pi}{8}}{\sin \frac{\pi}{8} * \cos \frac{\pi}{8}} \\
& =\frac{2 * \cos \frac{\pi}{4}}{\sin \frac{\pi}{4}}=2 \operatorname{ctg} \frac{\pi}{4}
\end{aligned}
$$

Hence, we have equality

$$
\operatorname{ctg} \frac{\pi}{8}-\frac{1}{\operatorname{tg} \frac{\pi}{8}}=\operatorname{ctg} \frac{\pi}{8}
$$

or

$$
\operatorname{ctg}^{2} \frac{\pi}{8}-2 \operatorname{ctg} \frac{\pi}{8}-1=0
$$

For $z=\operatorname{ctg} \frac{\pi}{8}$ we find the value of $z$ by solving the quadratic equation

$$
z^{2}-2 z-1=0, \quad \text { wyroznik } \Delta=8, \quad \text { wartosc } \quad z=\frac{2+\sqrt{8}}{2}=1+\sqrt{2}
$$

So $\operatorname{ctg} \frac{\pi}{8}=1+\sqrt{2}$ and the radius of the circle inscribed in a regular octagon

$$
r=\frac{1}{2} a * \operatorname{ctg} \frac{\pi}{8}=\underbrace{\frac{a}{2}(1+\sqrt{2})}_{r}
$$

Question 17.29 Find the perimeter and the area of regular hexagon $A B C D C E F G$ of the side length $|A B|=2$

Question 17.30 Given the radius $r=8$ of the regular octagon, calculate the radius $R$ of the circle of the regular octagon.

Question 17.31 Given the side $a=3 \mathrm{~cm}$, calculate the radius $r$ of the inscribed circle and the radius $R$ of the circle described on the regular octagon.

Question 17.32 Construct a fororem octagon with side $a=3 \mathrm{~cm}$ using a compasses and a ruler. Measure the radius $r$ of the circle inscribed in the octagon and the radius $R$ of the circle of the regular octagon.

Question 17.33 (i) Find the perimeter and area of the angle $\triangle A B C$ with the sides

$$
|A B|=10, \quad|B C|=8, \quad|A C|=6
$$

(ii) Check if the triangle $\triangle A B C$ is rectangular.

Question 17.34 Find the perimeter and area of the triangle $\triangle A B C$ with the sides

$$
|A B|=3, \quad|B C|=5, \quad|A C|=7
$$

using Heron's formula
Question 17.35 Calculate the perimeter and area of the parallelogram $A B C D$ with the sides

$$
|A B|=|C D|=4, \quad|B C|=|A D|=5
$$

using Heron's formula

Question 17.36 Calculate the perimeter and area of the regular hexagon $A B C D C E$ by the side length $|A B|=5$

Question 17.37 Plot a regular octagon with compasses and a ruler

## Chapter 18

## Geometry in the space $R^{3}$. Stereometry

In this chapter we cover the following topics:

1. Cartesian coordinate system.
2. Points and vectors in space.
3. Parametric line equation
4. Prisms and cuboids, volume and surface area
5. Pyramids, volume and surface area
6. Solids of rotation: cylinder, sphere, cone, volume and surface area.
7. Platonic solids

Among the figures in space $R^{3}$, we distinguish prisms, cuboids, pyramids, cones and a sphere.
Also among the figures in space, we distinguish regular and Platonic solids. Regular solids have all congruent sides. Platonic solids include regular tetrahedron, regular cube, regular octahedron, dodecahedron and regular icosahedron. Platonic solids were considered in ancient times, in the Academy of Plato (427-347, B.C.), to be ideal figures.

### 18.1 Points and vectors in Cartesian space $R^{3}$

The positions of points and vectors are defined in the Cartesian coordinates.

$$
R^{3}=\left\{\left(x_{1}, x_{2}, x_{2}\right):-\infty<x_{1}, x_{2}, x_{3}<\infty .\right\}
$$

The point

$$
A=\left(a_{1}, a_{2}, a_{3}\right)
$$

in the Cartesian space $R^{3}$ has coordinates

$$
x_{1}=a_{1}, \quad x_{2}=a_{2}, \quad x_{3}=a_{3} .
$$

The position of the point $A=\left(a_{1}, a_{2}, a_{3}\right)$, is on the graph below


The following operations are performed on the points

$$
A=\left(a_{1}, a_{2}, a_{3}\right) \quad \text { and } \quad B=\left(b_{1}, b_{2}, b_{3}\right)
$$

- The sum of points

$$
\begin{aligned}
A+B & =\left(a_{1}, a_{2}, a_{3}\right)+\left(b_{1}, b_{2}, b_{3}\right) \\
& =\left(a_{1}+b_{1}, a_{2}+b_{2}, a_{3}+b_{3}\right)
\end{aligned}
$$

is equal to the point

$$
C=A+B
$$

with its coordinates

$$
c_{1}=a_{1}+b_{1}, \quad c_{2}=a_{2}+b_{2}, \quad c_{3}=a_{3}+b_{3} .
$$

- The difference of points

$$
\begin{aligned}
A-B & =\left(a_{1}, a_{2}, a_{3}\right)-\left(b_{1}, b_{2}, b_{3}\right) \\
& =\left(a_{1}-b_{1}, a_{2}-b_{2}, a_{3}-b_{3}\right)
\end{aligned}
$$

is equal to the points

$$
C=A-B
$$

with its the coordinates

$$
c_{1}=a_{1}-b_{1}, \quad c_{2}=a_{2}-b_{2}, \quad c_{3}=a_{3}-b_{3} .
$$

- The product of a point by a number $t$ is equal to the point

$$
t * A=t *\left(a_{1}, a_{2}, a_{3}\right)=\left(t * a_{1}, t * a_{2}, t * a_{3}\right)
$$

with coordinates

$$
c_{1}=t * a_{1}, c_{2}=t * a_{2}, c_{3}=t * a_{3}
$$

Example 18.1 These points will be given $A=(2,-3.4)$ and $B=(2,-1.3)$. Calculate

$$
\text { (i) } A+B, \quad \text { (ii) } \quad a b, \quad \text { (iii) } \quad 2 * A+3 * B
$$

Solution. We calculate

$$
\begin{aligned}
(i) A+B & =(2,-3,4)+(2,-1,3) \\
& =(2+2,-3-1,4+3) \\
& =(4,-4,7) \\
\text { Answer : } A+B & =C, \quad C=(4,-4,7) \\
(i i) \quad A-B & =(2,-3,4)-(2,-1,3) \\
& =(2-2,-3-(-1), 4-3) \\
& =(0,-2,1) \\
\text { Answer : } A-B & =C, \quad C=(0,-2,1)
\end{aligned}
$$

$$
\text { (iii) } \begin{aligned}
2 * A+3 * B & =2 *(2,-3,4)+3 *(2,-1,3) \\
& =(2 * 2+3 * 2,2 *(-3)+3 *(-1), 2 * 4+3 * 3) \\
& =(10,-9,17)
\end{aligned}
$$

Answer: $2 * A+3 * B=C, \quad C=(10,-9,17)$.
Question 18.1 Consider points

$$
A=(3,2,-1), \quad B=(1,-1,2) .
$$

Calculate

$$
\text { (i) } A+B, \quad \text { (ii) } A-B, \quad \text { (iii) } \quad 3 * A+5 * B
$$

### 18.1.1 Vectors in the Cartesian space $R^{3}$

The vector $\overrightarrow{A B}$ with beginning at point

$$
A=\left(a_{1}, a_{2}, a_{3}\right)
$$

and with the end at point

$$
B=\left(b_{1}, b_{2}, b_{3}\right)
$$

has coordinates detrmined as difference of points. That is

$$
\overrightarrow{A B}=B-A=\left[b_{1}-a_{1}, b_{2}-a_{2}, b_{3}-a_{3}\right] .
$$

12 For example, the bound vector starting at $A=(0,1,3)$ and with the end at $B=(2,0,5)$ has the coordinates

$$
\overrightarrow{A B}=B-A=(2,0.5)-(0,1,3)=[2,-1.2]
$$

Adding Vectors. The sum of vectors

$$
\vec{v}=\left[v_{1}, v_{2}, v_{3}\right] \quad \text { and } \quad \vec{w}=\left[w_{1}, w_{2}, w_{3}\right]
$$

is equal to the vector

$$
\vec{Q}=\left[z_{1}, z_{2}, z_{3}\right]=\left[v_{1}+w_{1}, v_{2}+w_{2}, v_{3}+w_{3}\right]
$$

with the ccordinates

$$
z_{1}=v_{1}+w_{1}, \quad z_{2}=v_{2}+w_{2}, \quad z_{3}=v_{3}+w_{3} .
$$

Example 18.2 Calculate the sum of vectors

$$
\vec{v}=[1,2,1] \quad \text { and } \quad \vec{w}=[2,1,2]
$$

Solution. We calculate the sum

$$
\vec{v}+\vec{w}=[1,2,1]+(2,1,2)=[1+2,2+1,1+2]=[3,3,3]
$$

Answer. The sum of the given vectors $\vec{v}=[1,2,1] i \vec{w}=[2,1,2]$ is the vector $\vec{Q}=[3,3,3]$.
Difference of vectors. The difference of two vectors

$$
\vec{v}=\left[v_{1}, v_{2}, v_{3}\right] \quad \text { and } \quad \vec{w}=\left[w_{1}, w_{2}, w_{3}\right]
$$

is equal to the vector

$$
\vec{Q}=\left[z_{1}, z_{2}, z_{3}\right]=\vec{v}-\vec{w}=\left[v_{1}-w_{1}, v_{2}-w_{2}, v_{3}-w_{3}\right]
$$

with coordinates

$$
z_{1}=v_{1}-w_{1} \quad \text { and } \quad z_{2}=v_{2}-w_{2}, \quad z_{3}=v_{3}-w_{3} .
$$

[^27]Example 18.3 Calculate the difference of vectors

$$
\vec{v}=[1,2,6] \quad \text { and } \quad \vec{w}=[2,1,5]
$$

Solution. We calculate the difference of the vectors

$$
\vec{v}-\vec{w}=[1,2,6]-[2,1,5]=[1-2,2-1,6-5]=[-1,1,1]
$$

Answer. The result of subtracting the data of vectors $\vec{v}=[1,2,6]$ and $\vec{w}=[2,1,5]$ is the vector $\vec{Q}=[-1,1,1]$.

### 18.1.2 Scalar product of vectors in the Cartesian space $R^{3}$

${ }^{3}$ Scalar product of vectors is an important operation on vectors, which has wide in applications mathematics, physics, chemistry, and other sciences.

Definicja 18.1 The scalar product of the vectors $\vec{v}=\left[v_{1}, v_{2}, v_{3}\right] i \vec{w}=\left[w_{1}, w_{2}, w_{3}\right]$ is the number

$$
\begin{equation*}
(\vec{v}, \vec{w})=v_{1} * w_{1}+v_{2} * w_{2}+v_{3} * w_{3} \tag{18.1}
\end{equation*}
$$

So, the scalar product of vectors is not a vector, it is a number
Example 18.4 Calculate the scalar product of vectors

$$
\vec{v}=[2,5,3] \quad \text { and } \quad \vec{w}=[7,3,-2] .
$$

Solution. Using the formula (18.1) we calculate the scalar product of the given vectors.

$$
\begin{aligned}
(\vec{v}, \vec{w}) & =([2,5,3] *[7,3,-2]) \\
& =2 * 7+5 * 3+3 *(-2)=14+15-6=23 .
\end{aligned}
$$

Answer. The scalar product of the given vectors $\vec{v}=[2,5,3] i \vec{w}=[7,3,-2]$ is a number 23, we write

$$
(\vec{v}, \vec{w})=23
$$

The scalar product of vectors preserves all the properties of the arithmetic operation. Let's consider two vectors

$$
\vec{v}=\left[v_{1}, v_{2}, v_{3}\right] \quad \text { and } \quad \vec{w}=\left[w_{1}, w_{2}, w_{3}\right]
$$

- a scalar product is commutative

$$
(\vec{v}, \vec{w})=(\vec{w}, \vec{v})
$$

Indeed, we check that

$$
\begin{aligned}
(\vec{v}, \vec{w}) & =v_{1} * w_{1}+v_{2} * w_{2}+v_{3} * w_{3} \\
& =w_{1} * v_{1}+w_{2} * v_{2}+w_{3} * v_{3} \\
& =(\vec{w}, \vec{v})
\end{aligned}
$$

- scalar multiplication of vectors is separable from addition

$$
(\vec{v},(\vec{w}+\vec{Q}))=(\vec{v}, \vec{w})+(\vec{v}, \vec{Q})
$$

Indeed, we check that

$$
\begin{aligned}
(\vec{v}, \vec{w}+\vec{Q}) & =v_{1} *\left(w_{1}+z_{1}\right)+v_{2} *\left(w_{2}+z_{2}\right)+v_{3}\left(w_{3}+z_{3}\right) \\
& =v_{1} * w_{1}+v_{1} * z_{1}+v_{2} * w+2+v_{2} * z_{2}+v_{3} * w_{3}+v_{3} * z_{3} \\
& =\underbrace{v_{1} * w_{1}+v_{2} * w_{2}+v_{3} * z_{3}}_{(\vec{v}, \vec{w})}+\underbrace{v_{1} * z_{1}+v_{2} * z_{2}+v_{3} * z_{3}}_{(\vec{v}, \vec{Q})} \\
& =(\vec{v}, \vec{w})+(\vec{v}, \vec{Q})
\end{aligned}
$$

[^28]- The scalar product of the vector $\vec{v}$ by itself is equal to the square of its length

$$
\begin{aligned}
(\vec{v}, \vec{v}) & =v_{1} * v_{1}+v_{2} * v_{2}+v_{3} * v_{3} \\
& =v_{1}^{2}+v_{2}^{2}+v_{3}^{2}=|\vec{v}|^{2} .
\end{aligned}
$$

Now we will give an important theorem in the form of a sufficient and necessary condition
Theorem 18.1 .
Sufficient condition: If the scalar product is equal zero

$$
(\vec{v}, \vec{w})=0
$$

then the vectors $\vec{v}, \vec{w}$ are perpendicular, we write

$$
\vec{v} \perp \vec{w}
$$

Necessary condition: If the vectors $\vec{v}$ and $\vec{w}$ are perpendicular

$$
\vec{v} \perp \vec{w}
$$

then their clear product is equal to zero

$$
(\vec{v}, \vec{w})=0 .
$$

Together we write the necessary and sufficient condition in symbols

$$
\vec{v} \perp \vec{w} \Longleftrightarrow(\vec{v}, \vec{w})=0 .
$$

There is some evidence for this claim. Here we will give a proof based on the Pythagorean theorem. Namely, we will prove that a triangle with the arms $\vec{v}$ and $\vec{w}$ is rectangular if and only if the scalar product is equalo to zero

$$
(\vec{v}, \vec{w})=0
$$

Proof of the sufficient condition. We assume that the scalar product of the vectors $\vec{v}$ and $\vec{w}$ is equal to zero

$$
(\vec{v}, \vec{w})=0
$$

We will prove that the vectors $\vec{v}$ and $\vec{w}$ are perpendicular.
Calculate the square of the length of the vector difference $\vec{v} i \vec{w}$

$$
\begin{aligned}
|\vec{v}-\vec{w}|^{2} & =(\vec{v}-\vec{w}, \vec{v}-\vec{w}) \\
& =(\vec{v}, \vec{v})-2(\vec{v}, \vec{w})+(\vec{w}, \vec{w}) \\
& =|\vec{v}|^{2}-2(\vec{v}, \vec{w})+|\vec{w}|^{2}
\end{aligned}
$$

Note that if the scalar product

$$
(\vec{v}, \vec{w})=0
$$

Then length of sides of the triangle $\triangle A B C$

$$
\begin{aligned}
& \vec{w}|\vec{l}|=|\vec{v}|, \quad|A C|=|\vec{w}|, \quad|B C|=|\vec{v}-\vec{w}| \\
& \hline 0 \\
& |A B|=|\vec{v}|, \quad|A C|=|\vec{w}|, \quad|B C|=|\vec{v}-\vec{w}|
\end{aligned}
$$

hold the equality

$$
\begin{equation*}
|\vec{v}|^{2}+|\vec{w}|^{2}=|v \overrightarrow{-} w|^{2} \tag{18.2}
\end{equation*}
$$

On the other hand, the Pythagorean theorem shows that the sum of the squares of two sides of a triangle is equal to the square of the length of the third side (cf. (17.5) if and only if this triangle is rectangular.
Thus, the angle $\angle A C B$ between the vectors $\vec{v}$ and $\vec{w}$ is right angle if the scalar product of these vectors is zero.
End of proof of sufficient condition.
Proof of the necessary condition. Let vectors $\vec{v}$ and $\vec{w}$ be perpendicular.

$$
\vec{v} \perp \vec{w} .
$$

We will prove that the scalar product

$$
(\vec{v}, \vec{w})=0
$$

For this purpose, let's calculate the square of the length of the difference of the vectors $\vec{v}$ and $\vec{w}$.

$$
\begin{align*}
|\vec{v}-\vec{w}|^{2} & =(\vec{v}-\vec{w}, \vec{v}-\vec{w}) \\
& =(\vec{v}, \vec{v})-2(\vec{v}, \vec{w})+(\vec{w}, \vec{w})  \tag{18.3}\\
& =|\vec{v}|^{2}-2(\vec{v}, \vec{w})+|\vec{w}|^{2}
\end{align*}
$$

By assumptionle, the vectors $\vec{v}$ and $\vec{w}$ are perpendicular. So the sides of $A B$ and $A C$ of the triangle $\triangle A B C$ they are also perpendicular. Hence, the triangle $\triangle A B C$ is a right triangle
We know from the Pythagorean theorem that the square of the length of the hypotenuses $[B, C]$ is equal to the sum of the squares of the lengths of the legs $[A, B]$ and $[A, C]$, we write

$$
\begin{equation*}
|B C|^{2}=|A B|^{2}+|A C|^{2} \quad \text { or } \quad|\vec{v}-\vec{w}|^{2}=|\vec{v}|^{2}+|\vec{w}|^{2} \tag{18.4}
\end{equation*}
$$

where

$$
|A B|=|\vec{v}|, \quad|A C|=|\vec{w}|, \quad|B C|=|\vec{v}-\vec{w}| .
$$

The equality of (18.3) and (18.4) results in the equality of the sides

$$
\begin{aligned}
& |\vec{v}-\vec{w}|^{2}=|\vec{v}|^{2}-2(\vec{v}, \vec{w})+|\vec{w}|^{2} \\
& |\vec{v}-\vec{w}|^{2}=\left.\vec{v}\right|^{2}+|\vec{w}|^{2} \\
& |\vec{v}|^{2}-2(\vec{v}, \vec{w})+|\vec{w}|^{2}=\left.\vec{v}\right|^{2}+|\vec{w}|^{2}, \\
& -2(\vec{v}, \vec{w})=0
\end{aligned}
$$

So the scalar product of the vectors $\vec{v}$ and $\vec{w}$ is zero

$$
(\vec{v}, \vec{w})=0
$$

if $\vec{v} \perp \vec{w}$ are orthogonal. End of precondition proof. ${ }^{4}$
Example 18.5 Find the scalar product and the length of the vectors

$$
\vec{v}=[6,8,0], \quad \vec{w}=[9,12,0] .
$$

[^29]$$
\vec{v} \perp \vec{w} \Longleftrightarrow(\vec{v}, \vec{w})=0
$$

Solution. We calculate the scalar product using the formula (18.1) for the vectors

$$
\vec{v}=\left[v_{1}, v_{2}, v_{3}\right]=[6,8,0], \quad \text { and } \quad \vec{w}=\left[w_{1}, w_{2}, w_{3}\right]=[9,12,0]
$$

So the scalar product

$$
(\vec{v}, \vec{w})=6 * 9+8 * 12+0 * 0=54+96=150 .
$$

is equal 150 .
We know that the square of the length of the vector $\vec{v}=[6,8,0]$ is equal to the scalar product of this vector by itself.

$$
|\vec{v}|^{2}=(\vec{v}, \vec{v})=6 * 6+8 * 8+0 * 0=36+64=100
$$

Where the length of the vector

$$
|\vec{v}|=\sqrt{100}=10
$$

Similarly, we calculate the length of the vector $\vec{w}=[9,12,0]$

$$
\begin{aligned}
|\vec{w}| & =\sqrt{(\vec{w}, \vec{w})} \\
& =\sqrt{9 * 9+12 * 12+0 * 0} \\
& =\sqrt{81+144}=\sqrt{225}=15
\end{aligned}
$$

Example 18.6 For what value of the parameter $m$ vectors

$$
\vec{v}=[m, 6,3], \quad \vec{w}=[3,2,4] .
$$

are perpendicular ?
Solution. We calculate the scalar product using the formula (18.1) for the vectors

$$
\vec{v}=\left[v_{1}, v_{2}\right]=[m, 6,3], \quad \text { and } \quad \vec{w}=\left[w_{1}, w_{2}\right]=[3,2,4]
$$

Vectors are orthogonal if their scalar product is zero. We calculate the scalar product

$$
(\vec{v}, \vec{w})=m * 3+6 * 2+3 * 4=3 m+24=0
$$

Hence we get the equation

$$
3 m+24=0, \quad d l a \quad m=-\frac{24}{3}=-8
$$

In fact, we check that for $m=-8$ the scalar product of the vector $\vec{v}=[m, 6,3]$ by the vector $\vec{w}=[3,2,4]$ is equal to zero

$$
(\vec{v}, \vec{w})=-8 * 3+6 * 2+3 * 4=24-24=0
$$

Answer: Indeed the vectors

$$
\vec{v}=\left[v_{1}, v_{2}, v_{3}\right]=[m, 6,3], \quad \text { and } \quad \vec{w}=\left[w_{1}, w_{2}, w_{3}\right]=[3,2,4]
$$

are perpendicular for the value of parameter $m=-8$.
Question 18.2 Calculate the scalar product and the length of the vectors

$$
\vec{v}=[12,16,0], \quad \vec{w}=[15,20,0] .
$$

Question 18.3 For what value of the parameter $m$ vectors

$$
\vec{v}=[m, 15,2], \quad \vec{w}=[5,3,4] .
$$

are perpendicular?

### 18.1.3 The vector product

Let's consider two vectors

$$
\vec{v}=\left(v_{1}, v_{2}, v_{3}\right), \quad \text { and } \quad \vec{w}=\left(w_{1}, w_{2}, w_{3}\right)
$$

in three-dimensional space

$$
R^{3}=\left\{x=\left(x_{1}, x_{2}, x_{3}\right):-\infty<x_{1}, x_{2}, x_{3}<\infty\right\}
$$



The result of multiplication of the vector $\vec{v}$ by the vector $\vec{w}$ is the third vector $\vec{v} \times \vec{w}$, the coordinates of which are calculated from the Laplace expansion of the matrix formed from the coordinates vectors

$$
\left\{\begin{array}{lll}
1 & 1 & 1 \\
v_{1} & v_{2} & v_{3} \\
w_{1} & w_{2} & w_{3}
\end{array}\right\}
$$

ddocument Namely the product

$$
\vec{v} \times \vec{w}=\left[\operatorname{Det}\left\{\begin{array}{ll}
v_{2} & v_{3} \\
w_{2} & w_{3}
\end{array}\right\},-\operatorname{Det}\left(\left\{\begin{array}{ll}
v_{1} & v_{3} \\
w_{1} & w_{3}
\end{array}\right\}\right), \operatorname{Det}\left\{\begin{array}{ll}
v_{1} & v_{2} \\
w_{1} & w_{2}
\end{array}\right\}\right]
$$

Hence do we get the formula for the coordinates of the vector product

$$
\begin{equation*}
\vec{v} \times \vec{w}=\left[v_{2} * w_{3}-v_{3} * w_{2},-\left(v_{1} * w_{3}-v_{3} * w_{1}\right), v_{1} * w_{2}-v_{2} * w_{1}\right] . \tag{18.5}
\end{equation*}
$$

The vector $\vec{v} \times \vec{w}$ is perpendicular to vectors $\vec{v}$ and $\vec{w}$, and its square of length is equal to the area of the parallelogram with side with sides $\vec{v}$ and $\vec{w}$. So the length of the vector

$$
|\vec{v} \times \vec{w}|=\sqrt{\left|v_{2} * w_{3}-v_{3} * w_{2}\right|^{2}+\left|-\left(v_{1} * w_{3}-v_{3} * w_{1}\right)\right|^{2}+\left|v_{1} * w_{2}-v_{2} * w_{1}\right|^{2}}
$$

### 18.1.4 Area of the quadrilateral.

We will consider a quadrilateral $A B C D$

with vertices

$$
\begin{array}{ll}
A=\left(a_{1}, a_{2}, a_{3}\right), & B=\left(b_{1}, b_{2}, b_{3}\right) \\
C=\left(c_{1}, c_{2}, c_{3}\right), & D=\left(d_{1}, d_{2}, d_{3}\right)
\end{array}
$$

spanned on vectors

$$
\begin{array}{ll}
\vec{v}=\left[v_{1}, v_{2}, v_{3}\right]=\overrightarrow{A B}, & \vec{w}=\left[w_{1}, w_{2}, w_{3}\right]=\overrightarrow{A D} \\
\vec{q}=\left[z_{1}, z_{2}, z_{3}\right]=\overrightarrow{C B}, & \vec{t}=\left[t_{1}, t_{2}, t_{3}\right]=\overrightarrow{C D}
\end{array}
$$

where the coordinates of the vectors $\vec{v}, \vec{w}, \vec{q}, \vec{t}$ are determined by the differences in the coordinates of the vertices $A, B, C, D$ of the quadrilateralABCD

$$
\begin{array}{lll}
v_{1}=b_{1}-a_{1}, & v_{2}=b_{2}-a_{2}, & v_{3}=b_{3}-a_{3}, \\
w_{1}=d_{1}-a_{1}, & w_{2}=d_{2}-a_{2}, & w_{3}=d_{3}-a_{3}, \\
z_{1}=b_{1}-c_{1}, & z_{2}=b_{2}-c_{2}, & z_{3}=b_{3}-c_{3}, \\
t_{1}=d_{1}-c_{1}, & t_{2}=d_{2}-c_{2}, & t_{3}=d_{3}-c_{3} .
\end{array}
$$

Using the vector product (cf. (18.5)) we can calculate the area of any quadrilateral with the given coordinates of its vertices. Namely, the area of the quadrilateral $A B C D$ is equal to half the cave root of the vector product of vectors

$$
P_{A B C D}=\frac{1}{2} \vec{v} \times \vec{w}+\frac{1}{2} \vec{q} \times \vec{t}
$$

Example 18.7 Find the area of the quadrilateral $A B C D$ spanned on the vectors


We calculate the vector products of $\vec{v} \times \vec{w} i \vec{q} \times \vec{t}$ using the formulas(cf. (18.5))

$$
\begin{aligned}
\vec{v} \times \vec{w} & =[0 * 0-3 * 0,-(3 * 0-0 * 0), 3 * 3-0 * 0] \\
& =[0,0,9]
\end{aligned}
$$

and the product of vectors

$$
\begin{aligned}
\vec{q} \times \vec{t} & =[-6 * 0-3 * 0,-(0 * 0-3 * 0), 0 * 3-6 * 3] \\
& =[0,0,-18]
\end{aligned}
$$

We calculate the area of the quadrilateral $A B C D$ as the sum of half the square root squared from the vector product of the vectors $\vec{v}, \vec{w}$ and $\vec{q}, \vec{t}$.
Namely

$$
\begin{aligned}
P_{A B C D} & =\frac{1}{2}|\vec{v} \times \vec{w}|+\frac{1}{2}|\vec{q} \times \vec{t}| \\
& \left.=\frac{1}{2} \sqrt{0^{2}+0^{2}+9^{2}}+\frac{1}{2} \right\rvert\, \sqrt{0^{2}+0^{2}+(-18)^{2}} \\
& =\frac{1}{2} * 9+\frac{1}{2} * 18 \\
& =4.5+9=13.5
\end{aligned}
$$

5
Let us consider another example of calculating the area of a quadrilateral using a vector product.
Example 18.8 Find the area of the quadrilateral $A B C D$ spanned on the vectors

$$
\begin{aligned}
& \vec{v}=[4.0,0,0]=\overrightarrow{A B}, \\
& \vec{w}=[-0.5,2.4,0]=\overrightarrow{A D}, \\
& \vec{q}=[-1.0,4.2,0]=\overrightarrow{C B}, \\
& \vec{t}=[-3.5,-1.8,0]=\overrightarrow{C D}
\end{aligned}
$$



We calculate the vector products of $\vec{v} \times \vec{w}$ and $\vec{q} \times \vec{t}$ using the formulas (cf. (18.5))

$$
\begin{aligned}
\vec{v} \times \vec{w} & =[0 * 0-0 * 2.4,-(4 * 0+0.5 * 0), 4 * 2.4+0.5 * 0] \\
& =[0,0,9.6]
\end{aligned}
$$

[^30]and the product of vectors
\[

$$
\begin{aligned}
\vec{q} \times \vec{t} & =[4.2 * 0+1.8 * 0,-((-1) * 0-(-3.5) * 0), 1 * 1.8+4.2 * 3.5] \\
& =[0,0,16.5]
\end{aligned}
$$
\]

We calculate the area of the quadrilateral $A B C D$ as the sum of the half of the quartar from the vector product of the vectors $\vec{v}, \vec{w}$ and $\vec{q}, \vec{t}$.
Namely

$$
\begin{aligned}
P_{A B C D} & =\frac{1}{2}|\vec{v} \times \vec{w}|+\frac{1}{2}|\vec{q} \times \vec{t}| \\
& =\frac{1}{2} \sqrt{0^{2}+0^{2}+9.6^{2}}+\frac{1}{2} \sqrt{0^{2}+0^{2}+16.5^{2}} \\
& =\frac{1}{2} * 9.6+\frac{1}{2} * 16.5 \\
& =4.8+8.25=13.05
\end{aligned}
$$

### 18.1.5 Parametric equation of a line in space

Simple operations on points and vectors in space lead to a parametric equation of the straight line. Namely, let us consider two points

$$
A=\left(a_{1}, a_{2}, a_{3}\right), \quad B=\left(b_{1}, b_{2}, b_{3}\right),
$$

and vector

$$
\overrightarrow{A B}=B-A
$$

We can easily write the parametric equation of the line $L$

$$
L(t)=A+t * \overrightarrow{A B}
$$

or

$$
L(t)=(1-t) * A+B * t .
$$

Here the parameter is the real variable $t \in(-\infty, \infty)$

, Let us note that if $t$ changes from minus infinity $-\infty$ to plus infinity $\infty$, then the point $L(t)$ moves along the line L.
A parametric equation for the straight line, we also write in in terms of points $A$ and $B$. Since the
vector $\overrightarrow{A B}=B-A$, the parametric equation of the straight line going through the points $A$ and $B$ has the form

$$
\begin{aligned}
L(t) & =A+(B-A) * t \\
\text { or } \quad L(t) & =A+\overrightarrow{A B} * t \\
\text { or } \quad L(t) & =(1-t) * A+B * t, \quad-\infty<t<\infty .
\end{aligned}
$$

Example 18.9 Write a parametric straight line equation
(i) with the beginning at point $A=(1,2,-1)$ and the direction of the vector $\vec{v}=[2,-1,4]$
(ii) that goes through the points $A=(1,-1,2)$ and $B=(2,1,2)$

## Solution.

(i) Substitute the data:

1. point $A=(1,-1,2)$ and the vector $\vec{v}=(2,-1,4)$ for the general equation

$$
\begin{aligned}
L(t) & =A+t * \vec{v} \\
& =(1,-1,2)+t *(2,-1,4) \\
& =(1+2 t,-1-t, 2+4 t) .
\end{aligned}
$$

Answer. $L(t)=(1+2 t,-1-t, 2+4 t), \quad-\infty<t<\infty$.
(ii) Substitute the data: point $a=(1,-1,2)$ and point $B=(2,1,2)$ into the general equation

$$
\begin{aligned}
L(t) & =(1-t) A+t * B=(1-t)(1,-1,2)+t *(2,-1,4) \\
& =((1-t)+2 t,(1-t)-t, 2(1-t)+4 t) \\
& =(1+t, 1-2 t, 2+2 t)
\end{aligned}
$$

Answer. $L(t)=(1+t, 1-2 t, 2+2 t), \quad-\infty<t<\infty$.
Question 18.4 Write a parametric straight line equation
(i) starting at point $A=(0,1,-1)$ and vector direction $\vec{v}=[2,1,3]$
(ii) going through the points $A=(3,1,2)$ i $B=(0,2,2)$

### 18.2 Prism with the base of an equilateral triangle

A prism with vertices $A, B, C, D, E, F$ and the base of a eqilateral triangle $A B C$ with a side of the base a.


On the graph a prism with the base of a regular triangle $\triangle A B C$, the side lengths

$$
|A B|=|A C|=B C \mid=a
$$

and hight $h$.
The surface of the prism consists of two bases of lower base and upper base and of surfaces of the three side walls.
Thus the total area and volume of the prism we calculate from the formulas
The total area

$$
P_{c}=\frac{a^{2} \sqrt{3}}{2}+3 a * h .
$$

and the volume

$$
V=\left(\frac{a^{2} \sqrt{3}}{2}\right) * h
$$

Example 18.10 Consider a prism with the base of a regular triangle with the sides $a=4$, and the height of the prism $h=6$.
Calculate
(i) total area of the prism,
(ii) volume of the prism.

Solution. Substituting the data $a=4$ and $h=6$ into the formulas into the total area and volume, we calculate
(i) area of the total area of the prism

$$
P_{c}=\frac{a^{2} \sqrt{3}}{2}+3 a * h=\frac{4^{2} \sqrt{3}}{2}+3 * 4 * 6=8 \sqrt{3}+72
$$

(ii) the volume of the prism

$$
V=\left(\frac{a^{2} \sqrt{3}}{2}\right) * h=\left(\frac{4^{2} \sqrt{3}}{2}\right) * 6=48 \sqrt{3} .
$$

Question 18.5 Consider a prism with the base of a regular triangle with the sides $a=2$, and ofthe height of the prism $h=5$.
Calculate
(i) total area of the prism,
(ii) volume of the prism.

### 18.3 Cuboid with the base of a rectangle

Below on he grath cuboid with the base of the rectangle $A B C D$ with the length of the sides of the rectangle $|A B|=a,|B C|=b$ and of the height of the cuboid $|B F|=h$


The total area of the cuboid consists of two bases and four side walls.
So that total area of the cuboid and its volume are given by the formulas:

The total area

$$
P_{c}=2 a * b+2 a * h+2 b * h .
$$

and volume of the cuboid

$$
V=a * b * h
$$

Example 18.11 Consider cuboid with the base of a rectangle with dimensions $a=4, b=5$ and of hight $h=6$, compute
(i) total area of the cuboid,
(ii) volume of cuboid.

Solution. Substituting into formulas we calculate
(i) The total area

$$
P_{c}=2 a * b+2 a * h+2 b * h=2 * 4 * 5+2 * 4 * 6+2 * 5 * 6=148 .
$$

(ii) volume $V=a * b * h=4 * 5 * 6=120$.

Question 18.6 Consider a cuboid with the base of a rectangle with dimensions $a=2, b=3$ and of hight $h=5$, calculate
(i) total area of the cuboid,
(ii) volume of the cuboid.

### 18.3.1 Regular cube

$A$ regular cube is a cuboid of which all six faces walls are congruent and all adges equal a.
Below on the graph are marked 8 vertices $A, B, C, D, E, F, G, H, 12$ adges $[A, B ; b, C, \ldots,[G, H]$ and 2 diagonals $[A, C],[[E, G]$

Area $P_{c}=6 a^{2}$
Volume $V=a^{3}$
Diagonal of base $d_{b}=a \sqrt{2}$
$|[A, C]|=d_{b}=a \sqrt{2}$
Diagonal of cube $=d_{c}=a \sqrt{3}$
$|[A, G]|=d_{c}=a \sqrt{3}$


From the grath above we can easly find the following formullas


Example 18.12 Consider the cube with side $a=4$, calculate
(i) total area
(ii) volume.
(iii) diagonal of the base
(iv) diagonal of the cube.

Solution. Substituting into formulas we calculate
(i) area of total cube $P_{c}=6 a^{2}=6 * 4^{2}=96$,
(ii) volume of a cube $V_{c}=a^{3}=4^{3}=64$.
(iii) the diagonal of the cube base $d_{p}=a \sqrt{2}=4 \sqrt{2}$.
(iv) the diagonal of the cube $d=a \sqrt{3}=4 \sqrt{3}$.

Question 18.7 For a cube with side $a=5$, calculate
(i) the total area of the cube,
(ii) volume of the cube.
(iii) the diagonal of the cube base.
(iv) the diagonal of the 's face.

### 18.3.2 A prism with a base regular hexagon

The drawn of a regular prism with base of a regular chexagon we present below


The area of the base of this prism consists of the areas of 6 equilibrium triangles

$$
P_{b a s e}=6 * \frac{a^{2} \sqrt{3}}{4}
$$

Thus the area of the hexogan is eqoal

$$
P_{b a s e}=\frac{3 a^{2} \sqrt{3}}{2}
$$

Total area of a prism based on a regular hexagon

$$
\begin{aligned}
P_{c} & =2 P_{\text {base }}+6 * a * h=2 \frac{3 a^{2} \sqrt{3}}{2}+6 a * h, \\
P_{c} & =\underbrace{3 a^{2} \sqrt{3}+6 a * h}_{\text {area of prism }}
\end{aligned}
$$

The volume of a prism based on a hexagon of a regular cygon

$$
V=P_{\text {base }} * h=\underbrace{\frac{3 a^{2} \sqrt{3}}{2} * h}_{\text {volumeofprism }}
$$

Example 18.13 For a prism with the base of a regular hexagon with side a $=$ 2, high $h=4$, calculate
(i) total area,
(ii) volume.

Solution. Substituting into the formulas for the total area and volume, we calculate
(i) total area

$$
P_{c}=3 a^{2} \sqrt{3}+6 a * h=3 * 2^{2} \sqrt{3}+6 * 2 * 4=\underbrace{12 \sqrt{3}+48}_{\text {area }} .
$$

(ii) volume

$$
V=P_{l} * h=\frac{3 a^{2} \sqrt{3}}{2} * h=\frac{3 * 2^{2} \sqrt{3}}{2} * 4=\underbrace{6 \sqrt{3}}_{\text {volume }}
$$

Question 18.8 For a prism with the base of a regular hexagon with side $a=4$, height $h=5$, calculate
(i) total area
(ii) volume.

### 18.4 Pyramids

A pyramid is a polyhedron, the base of which is any polygon and the side walls are triangles with a common vertex. Among the pyramids, we distinguish regular pyramids, the base of which is a regular polygon and the bottom of the height lies in the center of the circle described at the base of the pyramid.

### 18.4.1 Regular tetrahedron

A regular tetrahedron has all four faces that are equilateral triangles. Thus, face angles are $60^{\circ}$ or $\frac{\pi}{3}$ radians. Area of each face $\frac{a^{2} \sqrt{3}}{4}$, where $a$ is the length of each edge of the tetrahedron. The total area of a regular tetrahedron is four times the area of one wall.

$$
P_{c}=a^{2} \sqrt{3} .
$$

We compute the l tetrahedron edge from the Pythagorean theorem. Namely, we know that the height of the sidewall $h=\frac{a \sqrt{3}}{2}$. Its saucer lies in the middle of the base edge $\frac{a}{2}$.
So we calculate

$$
l^{2}=h^{2}+\left(\frac{a}{2}\right)^{2}=\left(\frac{a \sqrt{3}}{2}\right)^{2}+\frac{a^{2}}{4}
$$

The volume of a formal tetrahedron is one third of the base area times the height of $H$

$$
V=\frac{1}{3} \frac{a^{2} \sqrt{3}}{4} * H
$$

The amount of $H$ is calculated depending on the given edge of $a$. Namely, the bottom of the height $h$ of the side wall lies at the intersection of the height of the base at a point distant from the vertex of the triangle by $\frac{2}{3} h=\frac{2}{3} * \frac{a \sqrt{3}}{2}$. Tetrahedron edge $l=a$. From the Pythagorean theorem, we calculate the height of the tetrahedron

$$
H^{2}=a^{2}-\left(\frac{2}{3} h\right)^{2}=a^{2}-\left(\frac{2}{3} * \frac{a \sqrt{3}}{2}\right)^{2}=\frac{2}{3} a^{2}, \quad H=a \sqrt{\frac{2}{3}} .
$$

So the volume of the tetrahedron

$$
\begin{aligned}
V & =\frac{1}{3} \frac{a^{2} \sqrt{3}}{4} * H \\
V & =\frac{1}{3} \frac{a^{2} \sqrt{3}}{4} * a \sqrt{\frac{2}{3}}=\frac{a^{3}}{12} \sqrt{2}
\end{aligned}
$$


a

$$
\text { Regular tetrahedron } \underbrace{P_{c}=a^{2} \sqrt{3}}_{\text {area }}, \quad \underbrace{V=\frac{a^{3}}{12} \sqrt{2}}_{\text {volume }}
$$

### 18.4.2 Pyramid with a square base

The drawn of a pyramid with square base we present below

$\underbrace{V=\frac{1}{3} a^{2} * H}_{\text {volume }}$

where:

- a the side of the square at the base of the pyramid
- $H$ the height of the pyramid
- $h$ the height of the side wall of the pyramid
- l the side edge of the pyramid
- $P_{a}$ the area of the base of the pyramid
- $P_{l}$ the lateral area of the pyramid
- $P_{c}$ the total area of the pyramid
- $V$ the volume of the pyramid

The area of the base of a regular pyramid is equal to the area of the square $P_{a}=a^{2}$ with the side $a$. The lateral surface area of the regular pyramid is equal to the area of four isosceles triangles with the base $a$ and the height $\mathbf{h}$.
So that

$$
P_{l}=4 \frac{a * h}{2} .
$$

Thus the lateral area of the pyramid is equal

$$
P_{l}=2 a * h
$$

The height of the side wall is expressed depending on the side a and the edge l. Namely, we calculate the height from the Pythagorean theorem

$$
h^{2}=l^{2}-\left(\frac{a}{2}\right)^{2}, \quad h=\sqrt{l^{2}-\frac{a^{2}}{4}}, \quad h=\frac{1}{2} \sqrt{4 l^{2}-a^{2}} .
$$

Then the side wall field

$$
P_{l}=\frac{1}{4} a * \sqrt{4 l^{2}-a^{2}} .
$$

The total area of the pyramid is equal the area of the base $P_{a}$ plus the lateral area $P_{l}$.
Total area of a regular pyramid

$$
\begin{aligned}
& P_{c}=P_{l}+P_{c} \\
& P_{c}=a^{2}+a \sqrt{4 l^{2}-a^{2}}
\end{aligned}
$$

The volume of the regular pyramid

$$
V=\frac{1}{3} a^{2} * H
$$

### 18.4.3 Regular pyramid with a hexagonal base

The drawn of a pyramid with hexagonal base we present below

Pyramid $P_{c}=\frac{3}{2}\left[a^{2} \sqrt{3}+a \sqrt{4 l^{2}-a^{2}}\right]$

Volume $V=\frac{1}{2} a^{2} \sqrt{3} * H$
where


- a the side of the hexagon in the base of the pyramid is up
- $H$ the height of the pyramid
- $h$ the height of the side wall of the pyramid
- l the side edge of the pyramid
- $P_{a}$ the area of the base of the pyramid
- $P_{0}$ the area of the side of the pyramid
- $P_{c}$ is the total area of the pyramid
- $V$ the volume of the pyramid

The area of the base of a regular pyramid is equal to the area $P_{a}$ of a regular hexagon with side $a$.
Thus the surface of the hexagon

$$
P_{a}=\frac{3 a^{2} \sqrt{3}}{2}
$$

The surface of side wall of a regular pyramid $P_{l}$ is equal to the area of six isosceles triangles with the base $a$ and the height $\mathbf{h}$. The area of the side of the pyramid $P_{0}$ is equal to the area of an isosceles triangle with the basis a and the height $h$.

$$
P_{0}=\frac{1}{2} a * h .
$$

We express the height of the side wall depending on the side a and the edge l. Namely, we calculate the height from the Pythagorean theorem

$$
h^{2}=l^{2}-\left(\frac{a}{2}\right)^{2}, \quad h=\sqrt{l^{2}-\frac{a^{2}}{4}}, \quad h=\frac{1}{2} \sqrt{4 l^{2}-a^{2}} .
$$

The side walls surface

$$
P_{0}=\frac{1}{4} a * \sqrt{4 l^{2}-a^{2}} .
$$

The area of the total area of the pyramid is equal to the area of a regular hexagon in the base with the side a plus the area six isosceles triangles with base a and arms $l$.
Total area of a regular pyramid

$$
\begin{aligned}
P_{c} & =P_{a}+6 P_{0} \\
P_{c} & =\frac{3 a^{2} \sqrt{3}}{2}+\frac{6}{4} a \sqrt{4 l^{2}-a^{2}} \\
P_{c} & =\underbrace{\frac{3}{2}\left[a^{2} \sqrt{3}+a \sqrt{\left.4 l^{2}-a^{2}\right]}\right.}_{\text {total surface }}
\end{aligned}
$$

The volume of the pyramid

$$
\begin{aligned}
V & =\frac{1}{3} \frac{3 a^{2} \sqrt{3}}{2} * H \\
V & =\frac{a^{2} \sqrt{3}}{2} * H
\end{aligned}
$$

### 18.5 Revolving solids

Among the revolving solids, we distinguish a cylinder, a cone and a ball.

### 18.5.1 Cylinder

A cylinder is created by rotating a rectangle around one of its sides. The simple shape of the cylinder leads to obvious formulas on its total area and volume.


The figure above shows the radius $R$ and the height $H$ of the cylinder with the diameter of the base $|A B|=2 R$ and the radius of the upper base $\left|O^{*} B^{*}\right|=R$. The letters $O$ and $O^{*}$ denote the centers of the circles at the lower and upper bases.
The total area of the cylinder consits with area of the lowre base $\pi R^{2}$ and with the area of upper base $\pi^{2} R^{2}$ and of the area of side wall $2 \pi * R * H$
Thus the total area of a cylinder

$$
\begin{aligned}
& P_{c}=2 \pi R^{2}+2 \pi * R * H \\
& P_{c}=2 \pi * R(R+H)
\end{aligned}
$$

The volume of a cylinder

$$
V=\pi R^{2} H
$$

### 18.5.2 Cone

A cone is created by rotating a right triangle around one of its legs.
The draw of a cone is presented below
where

$$
C
$$

- $R$ radius of the base of the cone
- l forming of a cone
- $H$ height of the cone
- diameter $|A B|=2 R$ of cone base
- center $O$ of the base of the cone with the apex of $C$
- $P_{l}$ lateral surface of the cone
- $P_{c}$ total area of the cone
- $V$ cone volume
- base area of the cone $P_{0}=\pi R^{2}$,
- cone lateral area $P_{l}=2 \pi R l$
- total area of the cone $P_{c}=\pi R(R+H)$
- cone volume $\frac{1}{3} \pi R^{2} H$.


### 18.5.3 Sphere

Let us consider the sphere with the center at point $O$ of radius $R$ drawn below


The area of sphere

$$
P=4 \pi R^{2}
$$

and volume $V=\frac{4}{3} \pi R^{3}$

Example 18.1 Calculate the area and volume of the sphere with a radius $R=5$.
Solution. Substituting $R=5$ into the formula for the surface of the sphere

$$
S=4 \pi R^{2}
$$

and to the formula for the volume of a sphere

$$
V=\frac{4}{3} \pi R^{3}
$$

We calculate the surface of the sphere

$$
S=4 \pi * 5^{2}=100 \pi
$$

and volume

$$
V=\frac{4}{3} \pi 5^{3}=\frac{500}{3} \pi
$$

## Chapter 19

## Trigonometry

The word trigonometry was used since ancient times in Babylon, Egypt and Greece. The term trigonometry covers the measurement relationships between sides and angles in triangles

### 19.1 Trigonometric functions

- $\sin \alpha$, read sine $\alpha, \quad \cos \alpha, \quad$ read cosine $\alpha$,
- $\operatorname{tg} \alpha$ or $\tan \alpha$, read the tangent $\alpha$,
- $\operatorname{ctg} \alpha$ or $\operatorname{ctg} \alpha$, read cotangence $\alpha$,
- $\sec \alpha$, read secant $\alpha, \csc \alpha, \quad$ read $\operatorname{cosec} a n t ~ \alpha$,
- $\sin h \alpha=\frac{e^{\alpha}-e^{-\alpha}}{2}$, read the hyperbolic sine $\alpha$,
- $\cos h \alpha=\frac{e^{\alpha}+e^{-\alpha}}{2}$, read the hyperbolic cosine $\alpha$

The trigonometric functions are defined in a right triangle or on a trigonometric circle.
Let us consider a rectangular triangle $\triangle A B C$ with the vertices $A, B, C$ and the perpendiculars sides $A C$ and $B C$ and the hypotenuse of $A B$.


[^31]The lengths of perpendicular legs $[A, B] \perp[B, C]$ and the hypotenuse satify the relations listed below

$$
a=|B C|, \quad b=|A C|, \quad c=|A B|
$$

Definicja 19.1 The sine of the angle $\alpha$ is the ratio of the cathetus a to the hypotenuse $c$

$$
\sin \alpha=\frac{a}{c}
$$

Definicja 19.2 Cosine of angle $\alpha$ is the ratio of cathetus $b$ to hypotenuse $c$

$$
\cos \alpha=\frac{b}{c}
$$

Definicja 19.3 The tangent of the angle $\alpha$ is the ratio of the side $a$ to the side $b$

$$
\operatorname{tg} \alpha=\frac{a}{b} \quad \text { or } \quad \tan \alpha=\frac{a}{b}
$$

Definicja 19.4 The cotangent of the angle $\alpha$ is the ratio of the side $b$ to the side $a$.

$$
\operatorname{ctg} \alpha=\frac{b}{a} \quad \text { or } \quad \operatorname{ctan} \alpha=\frac{b}{a}
$$

Definicja 19.5 The secant of the angle $\alpha$ is the inverse of the sine of the angle $\alpha$. Therefore

$$
\sec \alpha=\frac{c}{a}
$$

Definicja 19.6 The cosecant of $\alpha$ is the inverse of the cosine of $\alpha$. Therefore

$$
\csc \alpha=\frac{c}{b}
$$

Note that the inverse of the tangent $\alpha$ is equal to the cotangent of angle $\alpha$ and the inverse of the cotangent of angle $\alpha$ is equal to the tangent of angle ta $\alpha$

$$
\frac{1}{\operatorname{tg} \alpha}=\operatorname{ctg} \alpha, \quad \frac{1}{\operatorname{ctg} \alpha}=\operatorname{tg} \alpha
$$

Example 19.1 Give the values of trigonometric functions defined in a right triangle with sides $a=3, b=4, c=5$

Solution. Angle th of this right triangle $\alpha=30^{\circ}, \beta=60^{\circ}, \gamma=90^{\circ}$

$$
\begin{array}{ll}
\sin \alpha=\frac{3}{5}, & \cos \alpha=\frac{4}{5},
\end{array} \quad \operatorname{tg} \alpha=\frac{3}{4},
$$

Note that the definition of trigonometric functions in a right triangle applies only to angles

$$
0 \leq \alpha \leq 90^{\circ} \quad \text { or in the arc measure } \quad 0 \leq \alpha \leq \frac{\pi}{2}
$$

In a right triangle the angles $\alpha$ and $\beta$ change from 0 to right angle $\frac{\pi}{2}$. Therefore cotangent and cosecant are undefined for $\alpha=0$ and $\beta=0$. Also for $\alpha=\beta=\frac{\pi}{2}$ tangent and secant are not specified.
Also we consider the trigonometric functions on the trigonometric circle. Let us note that
sine and cosine functions are defined for all real values of the argument $\alpha \in\{-\infty, \infty\}$. The tangent functions are defined for the real values of the argument $\alpha \neq \frac{k \pi}{2}, k=0,1,2, \ldots$; and the cotangent function is defined for all real argument values $\alpha \neq k \pi, k=0,1,2,3, \ldots$;

The values of the sine and cosine functions belong to the closed interval $[-1,1]$. Values of the tangent and cotanges run the entire set of real numbers from minus infinite $-\infty$ to plus infinite $\infty$.

The sign of the values of the trigonometric functions depends on the first quadrant $I$, second II, third or fourth IV to which the argument $\alpha$ belongs.
To determine the sign of the value of trigonometric functions, we use the heuristic rule:

In the first quarter all are positive sine, cosine, tangent and cotangent, in the second only sine is positive, in the third tangent and cotangent are positive, and in the fourth only cosine is positive.

Question 19.1 Calculate the value of the trigonometric expression
(i) $\sin \frac{\pi}{6}+\cos \frac{\pi}{6}$
(ii) $\operatorname{tg} \frac{\pi}{6}+\operatorname{ctg} \frac{\pi}{6}$

Question 19.2 Calculate the value of the trigonometric expression
(i) $\sin \frac{4 \pi}{3}+\cos \frac{4 \pi}{3}$
(ii) $\operatorname{tg} \frac{4 \pi}{3}+\operatorname{ctg} \frac{4 \pi}{3}$

### 19.2 Trigonometric circle.

For all real, negative or positive angles, the trigonometric functions are defined in the trigonometric circle.



Definicja 19.7 The sine of the angle $\alpha$ is the ratio of the coordinate $\mathbf{y}_{\mathbf{1}}$ to the radius $\mathbf{R}$

$$
\sin \alpha=\frac{y_{1}}{R}
$$

Definicja 19.8 The cosine of the angle $\alpha$ is the ratio of the $\mathbf{x}_{\mathbf{1}}$ coordinate to the radius $\mathbf{R}$

$$
\cos \alpha=\frac{x_{1}}{R}
$$

Definicja 19.9 The tangent of $\alpha$ is the ratio of $\mathbf{y}_{\mathbf{1}}$ to $\mathbf{x}_{\mathbf{1}}$

$$
\operatorname{tg} \alpha=\frac{y_{1}}{x_{1}}, \quad x_{1} \neq 0
$$

Definicja 19.10 The cotangent of $\alpha$ is the ratio of the $\mathbf{x}_{\mathbf{1}}$ coordinate to the $\mathbf{y}_{\mathbf{1}}$ coordinate

$$
\operatorname{ctg} \alpha=\frac{x_{1}}{y_{1}}, \quad y_{1} \neq 0
$$

Definicja 19.11 The secant of the angle $\alpha$ is the inverse of the sine of the angle $\alpha$. Therefore

$$
\sec \alpha=\frac{R}{y_{1}}, \quad y_{1} \neq 0
$$

Definicja 19.12 The cosecant of $\alpha$ is the inverse of the cosine of $\alpha$. Therefor

$$
\csc \alpha=\frac{R}{x_{1}}, \quad x_{1} \neq 0
$$

In order to determine the values of trigonometric functions, it is enough to consider four trigonometric functions sine, cosine, tangent and cotangent, in the first quarter of trigonometric circle and then use the table of reduction formulas.

### 19.2.1 Reductive formulas

Directly from the definition of trigonometric functions, we observe that all functions are nonnegative in the first quadrant of the trigonometric circle, because for the angle

$$
0 \leq \alpha \leq 90^{\circ}
$$

the coordinates of the point $p=\left(x_{1}, y_{1}\right)$ are non-negative, that is $x_{1} \geq 0, y_{1} \geq 0$ and the radius $R>0$.
In the second quadrant, only the sine of $(\sin \alpha \geq 0)$, is non-negative because the coordinate $y_{1} \geq 0$.
In the third quadrant, tangent and cotanges $(\operatorname{tg} \alpha \geq 0, \operatorname{ctg} \alpha \geq 0)$, are non-negative, since both $x_{1} \leq 0, y_{1} \leq 0$ are negative and then the quotient of $\left(\frac{y_{1}}{x_{1}} \geq 0\right)$ or $\left(\frac{x_{1}}{y_{1}} \geq 0\right)$.
In the fourth quadrant, only the cosine of $(\cos \alpha \geq 0)$ is non-negative, since the coordinate $x_{1} \geq 0$. In this position $\alpha$, from the trigonometric graph we read the values of the trigonometric functions written in the table below

| $0 \leq \alpha \leq 90^{\circ}$ | $\sin \alpha \geq 0$ | $\cos \alpha \geq 0$ | $\operatorname{tg} \alpha \geq 0$ | $\operatorname{ctg} \alpha \geq 0$ |
| :--- | :---: | :---: | :---: | :---: |
| $90^{\circ} \leq \alpha \leq 180^{\circ}$ | $\sin \alpha \geq 0$ | $\cos \alpha \leq 0$ | $\operatorname{tg} \alpha \leq 0$ | $\operatorname{ctg} \alpha \leq 0$ |
| $180^{\circ} \leq \alpha \leq 270^{\circ}$ | $\sin \alpha \leq 0$ | $\cos \alpha \leq 0$ | $\operatorname{tg} \alpha \geq 0$ | $\operatorname{ctg} \alpha \geq 0$ |
| $270 \leq \alpha \leq 360^{\circ}$ | $\sin \alpha \leq 0$ | $\cos \alpha \geq 0$ | $\operatorname{tg} \alpha \leq 0$ | $\operatorname{ctg} \alpha \leq 0$ |

The trigonometric functions reach all possible absolute values in the first quadrant of the trigonometric circle. Thus, the other values differ only in sign. These differences set the reduction formulas, which are given below. First, notice that if the angle $0 \leq \alpha \leq 90^{\circ}$ is in the first quadrant then the angle $90^{\circ}-\alpha$ is also in the first quadrant and the angle $90^{\circ}+\alpha$ in the second quadrant. However, the angle $-\alpha$ lies in the fourth quadrant. In this position of the angle $\alpha$, from the trigonometric graph we read the values of the trigonometric functions written in the table below

| $\sin \left(90^{\circ}-\alpha\right)=\cos \alpha$ | $\sin \left(90^{\circ}+\alpha\right)=\cos \alpha$ | $\sin (-\alpha)=-\sin \alpha$ |
| :--- | :--- | :--- |
| $\cos \left(90^{\circ}-\alpha\right)=\sin \alpha$ | $\cos \left(90^{\circ}+\alpha\right)=-\sin \alpha$ | $\cos (-\alpha)=\cos \alpha$ |
| $\operatorname{tg}\left(90^{\circ}-\alpha\right)=\operatorname{ctg} \alpha$ | $\operatorname{tg}\left(90^{\circ}+\alpha\right)=-\operatorname{ctg} \alpha$ | $\operatorname{tg}(-\alpha)=-\operatorname{tg} \alpha$ |
| $\operatorname{ctg}\left(90^{\circ}-\alpha\right)=\operatorname{tg} \alpha$ | $\operatorname{ctg}\left(90^{\circ}+\alpha\right)=-\operatorname{tg} \alpha$ | $\operatorname{ctg}(-\alpha)=-\operatorname{ctg} \alpha$ |

Now, notice that if the angle $0 \leq \alpha \leq 90^{\circ}$ is in the first quadrant, then the angle $180^{\circ}-\alpha$ is in the second quadrant and the angle $180^{\circ}+\alpha$ is in the third quadrant.

| $\sin \left(180^{\circ}-\alpha\right)=\sin \alpha$ | $\sin \left(180^{\circ}+\alpha\right)=-\sin \alpha$ |
| :--- | :--- |
| $\cos \left(180^{\circ}-\alpha\right)=-\cos \alpha$ | $\cos \left(180^{\circ}+\alpha\right)=-\cos \alpha$ |
| $\operatorname{tg}\left(180^{\circ}-\alpha\right)=-\operatorname{tg} \alpha$ | $\operatorname{tg}\left(180^{\circ}+\alpha\right)=\operatorname{tg} \alpha$ |
| $\operatorname{ctg}\left(180^{\circ}-\alpha\right)=-\operatorname{ctg} \alpha$ | $\operatorname{ctg}\left(180^{\circ}+\alpha\right)=\operatorname{ctg} \alpha$ |

Note also that if the angle $0 \leq \alpha \leq 90^{\circ}$ lies in the first quadrant then the angle $270^{\circ}-\alpha$ is in the third quadrant and the angle $180^{\circ}+\alpha$ is in the fourth quadrant. Thus, we have the following reduction formulas:

| $\sin \left(270^{\circ}-\alpha\right)=-\cos \alpha$ | $\sin \left(270^{\circ}+\alpha\right)=-\cos \alpha$ |
| :--- | :--- |
| $\cos \left(270^{\circ}-\alpha\right)=-\sin \alpha$ | $\cos \left(270^{\circ}+\alpha\right)=\sin \alpha$ |
| $\operatorname{tg}\left(270^{\circ}-\alpha\right)=-\operatorname{tg} \alpha$ | $\operatorname{tg}\left(270^{\circ}+\alpha\right)=-\operatorname{ctg} \alpha$ |
| $\operatorname{ctg}\left(270^{\circ}-\alpha\right)=-\operatorname{ctg} \alpha$ | $\operatorname{ctg}\left(270^{\circ}+\alpha\right)=-\operatorname{tg} \alpha$ |

Below in the table, we present the collected reduction formulas in the arc measure of angles.

| angle | $\operatorname{sinus}$ | $\operatorname{cosinus}$ | $\operatorname{tangens}$ | $\operatorname{cotangens}$ |
| :--- | :--- | :--- | :--- | :--- |
| $\frac{\pi}{2}-\alpha$ | $\sin \left(\frac{\pi}{2}-\alpha\right)=\cos \alpha$ | $\cos \left(\frac{\pi}{2}-\alpha\right)=\sin \alpha$ | $\operatorname{tg}\left(\frac{\pi}{2}-\alpha\right)=\operatorname{ctg} \alpha$ | $\operatorname{ctg}\left(\frac{\pi}{2}-\alpha\right)=\operatorname{tg} \alpha$ |
| $\frac{\pi}{2}+\alpha$ | $\sin \left(\frac{\pi}{2}+\alpha\right)=\cos \alpha$ | $\cos \left(\frac{\pi}{2}+\alpha\right)=-\sin \alpha$ | $\operatorname{tg}\left(\frac{\pi}{2}+\alpha\right)=-\operatorname{ctg} \alpha$ | $\operatorname{ctg}\left(\frac{\pi}{2}+\alpha\right)=-\operatorname{tg} \alpha$ |
| $\pi-\alpha$ | $\sin (\pi-\alpha)=\sin \alpha$ | $(\cos \pi-\alpha)=-\cos \alpha$ | $\operatorname{tg}(\pi-\alpha)=-\operatorname{tg} \alpha$ | $\operatorname{ctg}(\pi-\alpha)=-\operatorname{ctg} \alpha$ |
| $\pi+\alpha$ | $\sin (\pi+\alpha)=-\sin \alpha$ | $\cos (\pi+\alpha)=-\cos \alpha$ | $\operatorname{tg}(\pi+\alpha)=\operatorname{tg} \alpha$ | $\operatorname{ctg}(\pi+\alpha)=\operatorname{tg} \alpha$ |
| $\frac{3 \pi}{2}-\alpha$ | $\sin \left(\frac{3 \pi}{2}-\alpha\right)=-\cos \alpha$ | $\cos \left(\frac{3 \pi}{2}-\alpha\right)=-\sin \alpha$ | $\operatorname{tg}\left(\frac{3 \pi}{2}-\alpha\right)=\operatorname{ctg} \alpha$ | $\operatorname{tg}\left(\frac{3 \pi}{2}-\alpha\right)=\operatorname{tg} \alpha$ |
| $\frac{3 \pi}{2}+\alpha$ | $\sin \left(\frac{3 \pi}{2}+\alpha\right)=-\cos \alpha$ | $\cos \left(\frac{3 \pi}{2}+\alpha\right)=\sin \alpha$ | $\operatorname{tg}\left(\frac{3 \pi}{2}+\alpha\right)=-\operatorname{ctg} \alpha$ | $\operatorname{ctg}\left(\frac{3 \pi}{2}+\alpha\right)=-\operatorname{tg} \alpha$ |
| $2 \pi-\alpha$ | $\sin (2 \pi-\alpha)=-\sin \alpha$ | $\cos (2 \pi-\alpha)=\cos \alpha$ | $\operatorname{tg}(2 \pi-\alpha)=-\operatorname{tg} \alpha$ | $\operatorname{ctg}(2 \pi-\alpha)=-\operatorname{ctg} \alpha$ |

### 19.3 Questions

Question 19.3 The side lengths of a right triangle $\triangle A B C$ are equal, respectively

$$
a=|B C|=6, \quad b=|A C|=8, \quad c=|A B|=10
$$

Calculate the values of the trigonometric functions

$$
\begin{array}{llll}
\sin \alpha, & \sin \beta, & \cos \alpha, & \cos \beta, \\
\operatorname{tg} \alpha, & \operatorname{tg} \beta, & \operatorname{ctg} \alpha, & \operatorname{ctg} \beta
\end{array}
$$

angles $\alpha, \beta$ opposite the respective sides of $B C, A C$.
Question 19.4 (i) Draw the positions of the points

$$
p=\left(p_{1}, p_{2}\right)=(\sqrt{3}, 1), \quad q=\left(q_{1}, q_{2}\right)=(-\sqrt{3},-1) .
$$

on trigonometric circle with radius $R=2$.
(ii) Calculate the values of trigonometric functions
(a) $\sin 30^{\circ}=\frac{p_{2}}{R}$
$\sin 60^{\circ}=\frac{p_{1}}{R}$
(b) $\cos 30^{\circ}=\frac{p_{1}}{R} \quad \cos 60^{\circ}=\frac{p_{2}}{R}$
(c) $\operatorname{tg} 30^{\circ}=\frac{p_{2}}{p_{1}} \quad \operatorname{tg} 60^{\circ}=\frac{p_{1}}{p_{2}}$
(d) $\operatorname{ctg} 30^{\circ}=\frac{p_{1}}{p_{2}} \quad \operatorname{ctg} 60^{\circ}=\frac{p_{2}}{p_{1}}$
(iii) Calculate the values of the trigonometric functions
(a) $\sin 210^{\circ}=\frac{q_{2}}{R}$
$\sin 240^{\circ}=\frac{q_{1}}{R}$
(b) $\cos 210^{\circ}=\frac{q_{1}}{R} \quad \cos 240^{\circ}=\frac{q_{2}}{R}$
(c) $\operatorname{tg} 210^{\circ}=\frac{q_{2}}{q_{1}} \quad \operatorname{tg} 240^{\circ}=\frac{q_{1}}{q_{2}}$
(d) $\operatorname{ctg} 210^{\circ}=\frac{q_{1}}{q_{2}} \quad \operatorname{ctg} 240^{\circ}=\frac{q_{2}}{q_{1}}$

Question 19.5 Using the reduction formulas, calculate the values of the trigonometric functions
(a) $\sin 120^{\circ}$
$\sin 150^{\circ}$
(b) $\cos 120^{\circ}$
$\cos 150^{\circ}$
(c) $\operatorname{tg} 120^{\circ}$
$\operatorname{tg} 150^{0}$
(d) $\operatorname{ctg} 120^{\circ}$
$\operatorname{ctg} 150^{0}$

Question 19.6 Using the reduction formulas, calculate the values of the trigonometric functions
(a) $\sin 210^{\circ}$
$\sin 240^{\circ}$
(b) $\cos 210^{\circ}$
$\cos 240^{\circ}$
(c) $\operatorname{tg} 210^{\circ}$
$\operatorname{tg} 240^{\circ}$
(d) $\operatorname{ctg} 210^{0}$
$\operatorname{ctg} 240^{0}$

Question 19.7 Using the reduction formulas, calculate the values of the trigonometric functions
(a) $\sin 300^{\circ}$
$\sin 330^{\circ}$
(b) $\cos 300^{\circ}$
$\cos 330^{0}$
(c) $\operatorname{tg} 300^{\circ}$
$\operatorname{tg} 330^{0}$
(d) $\operatorname{ctg} 300^{0}$
$\operatorname{ctg} 330^{\circ}$

Question 19.8 (i) Calculate the period of the function:
(a) $f(x)=\sin \frac{1}{3} x$,
(b) $f(x)=\cos \frac{1}{3} x$.
(c) $f(x)=\operatorname{tg} \frac{1}{3} x$,
(d) $f(x)=\operatorname{ctg} \frac{1}{3} x$.

Question 19.9 Draw a plot of the function
(i) $f(x)=\sin \frac{1}{3} x, \quad$ for $\quad 0 \leq x \leq 6 \pi$
(ii) $f(x)=\operatorname{tg} \frac{1}{3} x \quad$ for $\quad-3 \pi \leq x \leq 3 \pi$
(iii) $f(x)=\operatorname{ctg} \frac{4}{3} x \quad$ for $\quad 0 \leq x \leq \frac{3}{4} \pi$

### 19.3.1 Periodic functions

The function $f(x)$ is periodic if there is a positive number $\omega>0$ such that

$$
\begin{equation*}
f(x+\omega)=f(x) \tag{19.1}
\end{equation*}
$$

for each real value of the argument belonging to the domain $x \in D .{ }^{2}$
Clearly, if the function $f(x)$ is periodic with period $\omega>0$, then the following identity holds:

$$
f(x+k \omega)=f(x), \quad x \in D
$$

for every integer $k=0, \pm 1, \pm 2, \ldots$;
The period of the function $f(x)$ is the smallest of the numbers $\omega>0$ that satisfies the identity (19.1). ${ }^{3}$

[^32]We check below that trigonometric functions are periodic. Namely, we observe that if the radius $R$ rotates $360^{\circ}$ or in the arc by $2 \pi$, then the point $p=\left(x_{1}, y_{1}\right)$ will return to its original position. Moreover, if the radius $R$ turns in the positive or negative direction by a multiple of $\omega=360^{\circ}$ or in an arc measure by a multiple of $\omega=2 \pi$, then the point $p=\left(x_{1}, y_{1}\right)$ will also return to the starting position.
The period of the sine and cosine functions is the number $\omega=360^{\circ}$ or in the arc measure the number $\omega=2 \pi$. However, for the functions tangent and cotangent the period is the number $\omega=180^{\circ}$ or in the arc measure $\omega=\pi$. Indeed, the tangent and cotangent functions achieve the same values in the first and third quadrants of the trigonometric circle, since

$$
\operatorname{tg} \alpha=\frac{y_{1}}{x_{1}}=\frac{-y_{1}}{-x_{1}}, \quad \text { and } \quad \operatorname{ctg} \alpha=\frac{x_{1}}{y_{1}}=\frac{-x_{1}}{-y_{1}}, \quad x_{1} \neq 0, y_{1} \neq 0
$$

Example 19.2 Calculate the period of the function:

$$
f(x)=\sin \frac{3}{2} x
$$

Solution. We know that the sine function has the period $2 \pi$. Thus, the period of $f(x)$ is the number $\omega$ such that

$$
f(x+\omega)=\sin \frac{3}{2}(x+\omega)=\sin \left(\frac{3}{2} x+\frac{3}{2} \omega\right)=\sin \frac{3}{2} x=f(x)
$$

for every real one $x$.
Hence, we calculate the period

$$
\frac{3}{2} \omega=2 \pi, \quad \omega=\frac{4}{3} \pi
$$

We check that the period of $f(x)$ is the number $\omega=\frac{4}{3} \pi$. Indeed, we have equality

$$
\begin{aligned}
f(x+\omega)=f\left(x+\frac{4}{3} \pi\right) & =\sin \frac{3}{2}\left(x+\frac{4}{3} \pi\right) \\
& =\sin \left(\frac{3}{2} x+\frac{3}{2} \frac{4}{3} \pi\right) \\
& =\sin \left(\frac{3}{2} x+2 \pi\right)=\sin \frac{3}{2} x=f(x)
\end{aligned}
$$

### 19.3.2 Questions

Calculate the period of the function
Question 19.10
(i) $\sin \frac{\pi x}{2}$
(ii) $\cos \frac{\pi x}{2}$

Question 19.11 Provide a plot of the function
(i) $\sin \frac{\pi x}{4}, \quad-4 \pi \leq x \leq 4 \pi$
(ii) $\cos \frac{\pi x}{4}, \quad 0 \leq x \leq 8 \pi$.

Question 19.12 Calculate the period of the function
(i) $\sin \frac{2 \pi x}{3}$
(ii) $\cos \frac{2 \pi x}{3}$

Question 19.13 Provide a plot of the function

$$
\text { (i) } \sin \frac{2 \pi x}{3}, \quad-\frac{3 \pi}{2} \leq x \leq \frac{3 \pi}{2}
$$

(ii) $\cos \frac{2 \pi x}{3}, \quad 0 \leq x \leq 3 \pi$.

Question 19.14 Calculate the period of the function
(i) $\operatorname{tg} \frac{\pi x}{4}$
(ii) $\operatorname{ctg} \frac{\pi x}{4}$

### 19.3.3 Graphs of trigonometric functions

The trigonometric functions sine and cosine are periodic for all real values $x \in(-\infty, \infty)$. with the same period $\omega=2 \pi$.

These functions satisfy the identities

$$
\begin{aligned}
& f(x+2 \pi) \sin (x+2 \pi)=\sin x)=f(x) \\
& g(x+2 \pi)=\cos (x+2 \pi) \cos x=g(x)
\end{aligned}
$$

for everyone $x \in(-\infty, \infty)$.
When plotting trigonometric functions, we put the argument $x$ on axis as shown in the

the sine function

$$
|\sin x|=\left|\frac{y_{1}}{R}\right| \leq 1, \quad \text { because } \quad R \geq\left|y_{1}\right|, \quad \text { for } \quad-\infty<x<\infty
$$

we find that its values do not exceed the range $[-1,1]$. This means that for all the values of the argument $-\infty<x<\infty$

$$
-1 \leq \sin x \leq 1
$$

Indeed, we have the inequality

$$
|\sin x|=\left|\frac{y_{1}}{R}\right| \leq 1, \quad \text { when } \quad R \geq\left|y_{1}\right|, \quad \text { for } \quad-\infty<x<\infty
$$

Similarly, the cosine function is periodic with the period $2 \pi$ and defined for all real $x$ i.e. $-\infty<x<\infty$. Its values do not exceed the range [ -1.1 , Thus, the inequality holds

$$
|\cos x|=\left|\frac{x_{1}}{R}\right| \leq 1, \quad \text { when } \quad R \geq\left|x_{1}\right|, \quad \text { for } \quad-\infty<x<\infty
$$



The trigonometric functions tangent and cotangent are periodic with the period $\omega=\pi$. Indeed, the angle $x+\pi$ lies in the third quadrant of the trigonometric circle. From the table, read the value of $\operatorname{tg}(x+\pi)=\operatorname{tg} x$ we find that the following identity holds

$$
f(x+\pi)=\operatorname{tg}(x+\pi)=\operatorname{tg} x=f(x)
$$

for each argument $x$ in the domain of the tangent function


### 19.4 Trigonometric identities

A trigonometric identity is an equality that is true for all angle values in the domain of identity. Unlike identity, the trigonometric equation is only true for certain angle values in the domain of the equation.
Trigonometric formulas are the identities, which hold for all values of argument. As the example of trigonometric identity we present below "trigonometric one"

Example 19.3 The "trigonometric one" identity

$$
\sin ^{2} \alpha+\cos ^{2} \alpha=1
$$

holds for all real values of angle $\alpha \in(-\infty, \infty)$.
Straight from the definition of the sine and cosine functions, we find sides $a$ and $b$ of a right triangle $\triangle A B C$


From the Pythagorean theorem follows that the sum of the squares of the legs of the right triangle is equal to the square of the hypotenuse

$$
a^{2}+b^{2}=c^{2}
$$

After substitution

$$
a=c * \sin \alpha, \quad b=c * \cos \alpha
$$

we get identity

$$
(c \sin \alpha)^{2}+(c \cos \alpha)^{2}=c^{2}, c^{2}\left(\sin ^{2} \alpha+\cos ^{2} \alpha\right)=c^{2} \mid: c^{2}
$$

Hence follows the "trigonometric one: identity

$$
\sin ^{2} \alpha+\cos ^{2} \alpha=1
$$

for each value of $\alpha \in(-\infty, \infty)$.
From the "trigonometruc one" identity one can get the trigonometric identities listed below

$$
1+t g^{2} \alpha=\frac{1}{\cos ^{2} \alpha}
$$

In fact, from the definition of the tangent function, the identity equality follows

$$
1+\operatorname{tg}^{2} \alpha=1+\frac{\sin ^{2} \alpha}{\cos ^{2} \alpha}=\frac{\cos ^{2} \alpha+\sin ^{2} \alpha}{\cos ^{2} \alpha}=\frac{1}{\cos ^{2} \alpha}=\csc ^{2} \alpha
$$

for each angle $\alpha \neq(2 \pi+1) \frac{\pi}{2}, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots ;$ when $\cos \alpha \neq 0$.
4
Similarly the identity follows from the definiton of the cotangent function

$$
1+\operatorname{ctg}^{2} \alpha=1+\frac{\cos ^{2} \alpha}{\sin ^{2} \alpha}=\frac{\sin ^{2} \alpha+\cos ^{2} \alpha}{\sin ^{2} \alpha}=\frac{1}{\sin ^{2} \alpha}=\sec ^{2} \alpha
$$

for each angle $\alpha \neq k * \pi, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots ;$ when $\sin \alpha \neq 0 .{ }^{5}$

### 19.4.1 Questions

Question 19.15 Check the identity

$$
\sin ^{4} x-\cos ^{4} x=1-2 \cos ^{2} x \quad \infty<x<\infty
$$

Question 19.16 Check the identity

$$
\left(1+t g^{2} x\right) \cos ^{2} x=1 \quad x \neq \frac{k \pi}{2}, \quad k=0, \pm 1, \pm 2, \ldots
$$

Question 19.17 Check the identity

$$
\frac{1+\operatorname{tg}^{2} x}{1+\operatorname{ctg}^{2} x}=\operatorname{tg}^{2} x \quad x \neq \frac{k \pi}{2}, \quad k=0, \pm 1, \pm 2, \ldots
$$

Question 19.18 Prove the formula

$$
\sin (\alpha+\beta)+\sin (\alpha+\beta)=2 \sin \alpha \cos \beta
$$

for all values of angles $\alpha i \beta$.

### 19.4.2 Sine and cosine functions of sum and difference of angles $\alpha, \beta$

We will derive the following formulas for the sum and difference of two angles below

$$
\begin{align*}
& \sin (\alpha+\beta)=\sin \alpha \cos \beta+\sin \beta \cos \alpha, \\
& \sin (\alpha-\beta)=\sin \alpha \cos \beta-\sin \beta \cos \alpha, \\
& \cos (\alpha+\beta)=\cos \alpha \cos \beta-\sin \beta \sin \alpha,  \tag{19.2}\\
& \cos (\alpha-\beta)=\cos \alpha \cos \beta+\sin \beta \sin \alpha,
\end{align*}
$$

[^33]Let us consider the graph of right triangle $C$


Height $h$ of the triangle $\triangle A B C$

Let us observe that

$$
\begin{aligned}
& \sin \alpha=\frac{|A D|}{|A C|}, \quad \sin \beta=\frac{|D B|}{|B C|} \\
& \cos \alpha=\frac{h}{|A C|}, \quad \cos \beta=\frac{h}{|B C|} \\
& h=|A C| \cos \alpha,
\end{aligned} \quad h=|B C| \cos \beta-1 .
$$

The $P$ field of the triangle $\triangle A B C$ is the sum of the $P_{1}$ field of the $\triangle A D C$ field and the $P_{2}$ field of the triangle $\triangle D B C$

$$
\begin{equation*}
P=P_{1}+P_{2}=\frac{1}{2}|A C||B C| \sin ((\alpha+\beta) \tag{19.3}
\end{equation*}
$$

On the other hand, we know that

$$
\begin{equation*}
P_{1}=\frac{1}{2}|A C| h \sin \alpha, \quad P_{2}=\frac{1}{2}|B C| h \sin \beta, \tag{19.4}
\end{equation*}
$$

Comparing the fields defined by the equations (19.3) and (19.4) we get the sine formula of the sum of two angles $\alpha$ and $\beta$

$$
\begin{aligned}
& \frac{1}{2}|A C||B C| \sin \left((\alpha+\beta)=\frac{1}{2}|A C| h \sin \alpha+\frac{1}{2}|B C| h \sin \beta\right. \\
& |A C||B C| \sin (\alpha+\beta)=|A C||B C| \cos \beta \sin \alpha+|A C||B C| \cos \alpha
\end{aligned}
$$

Hence sinus sum

$$
\sin (\alpha+\beta)=\underbrace{\sin \alpha \cos \beta+\sin \beta \cos \alpha}_{\sin (\alpha+\beta)}
$$

We derive the remaining formulas using reduction formulas.

$$
\begin{aligned}
\sin ((\alpha-\beta)=\sin (\alpha+(-\beta)) & =\sin \alpha \cos (-\beta)+\sin (-\beta) \cos \alpha \\
& =\underbrace{\sin \alpha \cos \beta-\sin \beta \cos \alpha}_{\sin (\alpha-\beta)} \\
\cos (\alpha+\beta)=\sin \left(90^{\circ}-(\alpha+\beta)\right) & =\sin \left(\left(90^{\circ}-\alpha\right)-\beta\right) \\
& =\sin \left(90^{\circ}-\alpha\right) \cos \beta-\sin \beta \cos \left(90^{\circ}-\alpha\right) \\
& =\underbrace{\cos \alpha \cos \beta-\sin \alpha \sin \beta}_{\cos (\text { alpha+ } \beta)}
\end{aligned}
$$

$$
\begin{aligned}
\cos (\alpha-\beta)=\sin \left(90^{\circ}-(\alpha-\beta)\right) & =\sin \left(\left(90^{\circ}-\alpha\right)+\beta\right) \\
& =\sin \left(90^{\circ}-\alpha\right) \cos \beta+\sin \beta \cos \left(90^{\circ}-\alpha\right) \\
& =\underbrace{\cos \alpha \cos \beta+\sin \alpha \sin \beta}_{\cos (\alpha-\beta)}
\end{aligned}
$$

The tangent and cotangent formulas of the sum and difference of two angles result directly from the above formulas

$$
\begin{aligned}
& \operatorname{tg}(\alpha+\beta)=\frac{\sin (\alpha+\beta)}{\cos (\alpha+\beta)}=\frac{\sin \alpha \cos \beta+\sin \beta \cos \alpha}{\cos \alpha \cos \beta-\sin \beta \sin \alpha}=\underbrace{\frac{\operatorname{tg} \alpha+\operatorname{tg} \beta}{1-\operatorname{tg} \alpha \operatorname{tg} \beta}}_{\operatorname{tg}(\alpha+\beta)} \\
& \text { for } \quad \alpha+\beta \neq(2 k+1) \frac{\pi}{2}, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots ; \\
& \operatorname{ctg}(\alpha+\beta)=\frac{\cos (\alpha+\beta)}{\sin (\alpha+\beta)}=\frac{\cos \alpha \cos \beta-\sin \beta \sin \alpha}{\sin \alpha \cos \beta+\sin \beta \cos \alpha}=\underbrace{\frac{\operatorname{ctg} \alpha \operatorname{ctg} \beta-1}{\operatorname{ctg} \alpha+\operatorname{ctg} \beta}}_{c t g(\alpha+\beta)} \\
& \text { for } \quad \alpha+\beta \neq k \pi, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots ;
\end{aligned}
$$

Similarly, we derive the formulas for the tangent and cotangent of the difference of two angles

$$
\begin{aligned}
& \operatorname{tg}(\alpha-\beta)=\frac{\sin (\alpha-\beta)}{\cos (\alpha-\beta)}=\frac{\sin \alpha \cos \beta-\sin \beta \cos \alpha}{\cos \alpha \cos \beta+\sin \beta \sin \alpha}=\underbrace{\frac{\operatorname{tg} \alpha-\operatorname{tg} \beta}{1+\operatorname{tg} \alpha \operatorname{tg} \beta}}_{\operatorname{tg}(\alpha-\beta)} \\
& \text { for } \quad \alpha-\beta \neq(2 k+1) \frac{\pi}{2}, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots ; \\
& \operatorname{ctg}(\alpha-\beta)=\frac{\cos (\alpha-\beta)}{\sin (\alpha-\beta)}=\frac{\cos \alpha \cos \beta+\sin \beta \sin \alpha}{\sin \alpha \cos \beta-\sin \beta \cos \alpha}=\underbrace{\frac{\operatorname{ctg} \alpha \operatorname{ctg} \beta+1}{\operatorname{ctg} \beta-\operatorname{ctg} \alpha}}_{c t g(\alpha-\beta)} \\
& \text { for } \quad \alpha-\beta \neq k \pi, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots ;
\end{aligned}
$$

### 19.4.3 Double angle patterns

The double angle patterns follow directly from the sum formulas above, when $\alpha=\beta$

$$
\begin{aligned}
& \sin 2 \alpha=2 \sin \alpha \cos \alpha, \quad \text { for } \alpha \in(-\infty, \infty) \\
& \cos 2 \alpha=\cos ^{2} \alpha-\sin ^{2} \alpha, \quad \text { for } \alpha \in(-\infty, \infty) \\
& \operatorname{tg} 2 \alpha=\frac{2 \operatorname{tg} \alpha}{1-\operatorname{tg}^{2} \alpha}, \quad \text { for } \quad \alpha \neq(2 k+1) \frac{\pi}{4}, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots \\
& \operatorname{ctg} 2 \alpha=\frac{\operatorname{ctg}^{2} \alpha-1}{2 \operatorname{ctg}^{2} \alpha}, \quad \text { for } \quad \alpha \neq k \pi, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots
\end{aligned}
$$

### 19.4.4 Half angle formulas

The half angle formulas are obtained by substituting instead $\alpha$ half angle $\frac{1}{2} \alpha$ into the above formulas, then we get

$$
\begin{aligned}
& \sin \alpha=2 \sin \frac{1}{2} \alpha \cos \frac{1}{2} \alpha, \quad \alpha \in(-\infty, \infty) \\
& \cos \alpha=\cos ^{2} \frac{1}{2} \alpha-\sin ^{2} \frac{1}{2} \alpha, \quad \cos \alpha=1-2 \sin ^{2} \frac{1}{2} \alpha, \quad \cos \alpha=2 \cos ^{2} \frac{1}{2} \alpha-1 \\
& \quad \alpha \in(-\infty, \infty) \\
& \operatorname{tg} \alpha=\frac{2 \operatorname{tg} \frac{1}{2} \alpha}{1-\operatorname{tg}^{2} \frac{1}{2} \alpha} \text { for } \alpha \neq(2 k+1) \frac{\pi}{2}, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots \\
& \operatorname{ctg} \alpha=\frac{\operatorname{ctg}^{2} \frac{1}{2} \alpha-1}{2 \operatorname{ctg}^{2} \frac{1}{2} \alpha}, \quad \text { for } \quad \alpha \neq k \pi, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots
\end{aligned}
$$

### 19.4.5 Half of an angle trigonometric functions

The half-angle formulas are directly derived from the above half of the angle formulas. Namely, calculating cosine and sine from the formulas

$$
\cos \alpha=2 \cos ^{2} \frac{1}{2} \alpha-1, \quad \cos \alpha=1-2 \sin ^{2} \frac{1}{2} \alpha
$$

we get cosine and sine formulas for half of the angle $\alpha$

$$
\left|\cos \frac{1}{2} \alpha\right|=\sqrt{\frac{1+\cos \alpha}{2}}, \quad\left|\sin \frac{1}{2} \alpha\right|=\sqrt{\frac{1-\cos \alpha}{2}}
$$

for $\alpha \in(-\infty, \infty)$.
Half of an angle formulas for tangent and cotangent result directly from the definition of these functions and formulas for sine and cosine

$$
\left|\operatorname{tg} \frac{1}{2} \alpha\right|=\left|\frac{\sin \frac{1}{2} \alpha}{\cos \frac{1}{2} \alpha}\right|=\frac{\sqrt{\frac{1-\cos \alpha}{2}}}{\sqrt{\frac{1+\cos \alpha}{2}}}=\sqrt{\frac{1-\cos \alpha}{1+\cos \alpha}}
$$

for $\quad \alpha \neq(2 k+1) \pi, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots$;
The cotangent is the inverse of the tangent, therefore

$$
\operatorname{ctg} \frac{1}{2} \alpha=\sqrt{\frac{1+\cos \alpha}{1-\cos \alpha}}
$$

for $\quad \alpha \neq 2 k \pi, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots ;$

### 19.4.6 Expression of trigonometric functions by $\operatorname{tg} \frac{1}{2} \alpha$

Let us denote by

$$
t=\operatorname{tg} \frac{1}{2} \alpha \quad \text { for } \quad \alpha \neq(2 k+1) \pi, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots ;
$$

Then the trigonometric functions of the angle $\alpha$ can be written in the form of rational measurements of the variable $t$.

$$
\begin{array}{ll}
\sin \alpha=\frac{2 t}{1+t^{2}}, & \cos \alpha=\frac{1-t^{2}}{1+t^{2}}, \quad-\infty<t<\infty \\
\operatorname{tg} \alpha=\frac{2 t}{1-t^{2}}, \quad t \neq-1,1, & \operatorname{ctg} \alpha=\frac{1-t^{2}}{2 t} \quad t \neq 0
\end{array}
$$

Indeed, we know that

$$
\begin{aligned}
& \sin \alpha=2 \sin \frac{1}{2} \alpha \cos \frac{1}{2} \alpha \\
&=\frac{2 \sin \frac{1}{2} \alpha \cos \frac{1}{2} \alpha}{\sin ^{2} \frac{1}{2} \alpha+\cos ^{2} \frac{1}{2} \alpha} \\
&=\frac{2 \sin \frac{1}{2} \alpha \cos \frac{1}{2} \alpha}{\cos ^{2} \frac{1}{2} \alpha} \\
& \frac{\sin ^{2} \frac{1}{2} \alpha+\cos ^{2} \frac{1}{2} \alpha}{\cos ^{2} \frac{1}{2} \alpha}=\frac{2 t}{1+t^{2}}
\end{aligned}
$$

Similarly, the cosine function

$$
\begin{aligned}
\cos \alpha & =\cos ^{2} \frac{1}{2} \alpha-\sin ^{2} \frac{1}{2} \alpha \\
& =\frac{\cos ^{2} \frac{1}{2} \alpha-\sin ^{2} \frac{1}{2} \alpha}{\cos ^{2} \frac{1}{2}+\sin ^{2} \frac{1}{2} \alpha} \\
& =\frac{\frac{\cos ^{2} \frac{1}{2} \alpha-\sin ^{2} \frac{1}{2} \alpha}{\cos ^{2} \frac{1}{2} \alpha}}{\frac{\cos ^{2} \frac{1}{2}+\sin ^{2} \frac{1}{2} \alpha}{\cos ^{2} \frac{1}{2} \alpha}}=\frac{1-t^{2}}{1+t^{2}}
\end{aligned}
$$

For the tangent and cotangent functions, the angle fawn formulas result directly from their definition and the above-mentioned formulas for the sine and cosine functions

$$
\operatorname{tg} \alpha=\frac{\sin \alpha}{\cos \alpha}=\frac{\frac{2 t}{1+t^{2}}}{\frac{1-t^{2}}{1+t^{2}}}=\frac{2 t}{1-t^{2}}, \quad t \neq-1,1
$$

Cotanges is the inverse of Tangnsa. So the formula for the cotangent

$$
\operatorname{ctg} \alpha=\frac{1-t^{2}}{2 t}, \quad t \neq 0
$$

### 19.4.7 Sum and difference of trigonometric functions

Here are the following formulas for the sum and difference of trigonometric functions

$$
\begin{align*}
& \sin \alpha+\sin \beta=2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} \\
& \sin \alpha-\sin \beta=2 \sin \frac{\alpha-\beta}{2} \cos \frac{\alpha+\beta}{2} \\
& \cos \alpha+\cos \beta=2 \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} \\
& \cos \alpha-\cos \beta=-2 \sin \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2} \\
& \operatorname{tg} \alpha+\operatorname{tg} \beta=\frac{\sin (\alpha+\beta)}{\cos \alpha \cos \beta}  \tag{19.5}\\
& \operatorname{tg} \alpha-\operatorname{tg} \beta=\frac{\sin (\alpha-\beta)}{\cos \alpha \cos \beta} \\
& \operatorname{ctg} \alpha+\operatorname{ctg} \beta=\frac{\sin (\alpha+\beta)}{\sin \alpha \sin \beta} \\
& \operatorname{ctg} \alpha-\operatorname{ctg} \beta=\frac{\sin (\alpha-\beta)}{\sin \alpha \sin \beta}
\end{align*}
$$

The above formulas are given by the formulas (19.4) for the sine and cosine of the sum and the difference of angles. Namely, we introduce new variables

$$
\alpha=x+y, \quad \beta=x-y, \quad x=\frac{\alpha+\beta}{2}, \quad y=\frac{\alpha-\beta}{2}
$$

Using the formulas (19.4) for the sine and cosine of the sum and the difference of angles we notice that

$$
\begin{aligned}
\sin \alpha+\sin \beta & =\sin (x+y)+\sin (x-y) \\
& =(\sin x \cos y+\sin y \cos x)+(\sin x \cos y-\sin y \cos x) \\
& =2 \sin x \cos y=2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} \\
\sin \alpha-\sin \beta & =\sin (x+y)-\sin (x-y) \\
& =(\sin x \cos y+\sin y \cos x)-(\sin x \cos y-\sin y \cos x) \\
& =2 \sin y \cos x=2 \sin \frac{\alpha-\beta}{2} \cos \frac{\alpha+\beta}{2} \\
\cos \alpha+\cos \beta & =\cos (x+y)+\cos (x-y) \\
& =(\cos x \cos y-\sin x \sin y)+(\cos x \cos y+\sin x \sin y) \\
& =2 \cos x \cos y=2 \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} \\
\cos \alpha-\cos \beta & =\cos (x+y)-\cos (x-y) \\
& =(\cos x \cos y-\sin x \sin y)-(\cos x \cos y+\sin x \sin y) \\
& =-2 \sin x \sin y=-2 \sin \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2}
\end{aligned}
$$

The formulas of the sum and difference of the tangent and the cotangent result directly from the definition of the above formulas for the sine of cosine.

$$
\begin{align*}
\operatorname{tg} \alpha+\operatorname{tg} \beta & =\frac{\sin \alpha}{\cos \alpha}+\frac{\sin \beta}{\cos \beta}, \\
& =\frac{\sin \alpha \cos \beta+\sin \beta \cos \alpha}{\cos \alpha \cos \beta}=\frac{\sin (\alpha+\beta)}{\cos \alpha \cos \beta}  \tag{19.6}\\
\operatorname{tg} \alpha-\operatorname{tg} \beta & =\frac{\sin \alpha}{\cos \alpha}-\frac{\sin \beta}{\cos \beta}, \\
& =\frac{\sin \alpha \cos \beta-\sin \beta \cos \alpha}{\cos \alpha \cos \beta}=\frac{\sin (\alpha-\beta)}{\cos \alpha \cos \beta}
\end{align*}
$$

Similarly, we introduce the formula for the sum and difference of cotangents

$$
\begin{aligned}
& \operatorname{ctg} \alpha+\operatorname{ctg} \beta=\frac{\sin (\alpha+\beta)}{\sin \alpha \sin \beta} \\
& \operatorname{ctg} \alpha-\operatorname{ctg} \beta=\frac{\sin (\alpha-\beta)}{\sin \alpha \sin \beta}
\end{aligned}
$$

### 19.5 Trigonometric equations

Let's start with the simplest trigonometric equations, the solutions of which are part of solutions to more complex equations.

Example 19.4 Find all solutions of the equation

$$
\text { (i) } \quad \sin x=0, \quad \text { (ii) } \quad|\sin x|=1
$$

Solution (i). The main roots of this equation, i.e. the zeros of the sine function in its period from 0 to $360^{\circ}$ or in the arc measure in the range $0 \leq \alpha \leq 2 \pi$ are the solutions

$$
x=0, \quad \text { or } \quad x=\pi
$$

These solutions are marked on the graph $y=\sin x$.
 the following form:

$$
x_{k}=2 k \pi, \quad \text { or } \quad x_{k}=\pi+2 k \pi=(2 k+1) \pi,
$$

for integer $k$. Then all solutions are multiples of the number $\pi$,

$$
x_{k}=k \pi, \quad k=0, \pm 1, \pm 2, \ldots ;
$$

Solution (ii). Principal roots of the equation

$$
|\sin x|=1, \quad \text { or } \quad \sin x=1 \quad \text { or } \quad \sin x=-1
$$

are real numbers

$$
x=\frac{\pi}{2}, \quad \text { or } \quad x=\frac{3 \pi}{2} .
$$

We can get all the solution by adding to the principal roots a multiple of the period $2 \pi$. Thus, all solutions have the following form:

$$
x_{k}=\frac{\pi}{2}+2 k \pi, \quad \text { or } \quad x_{k}=\frac{3 \pi}{2}+\pi+2 k \pi
$$

or integer $k$. This means that all solutions are of the form:

$$
x_{k}=\frac{\pi}{2}+k \pi, \quad k=0, \pm 1, \pm 2, \ldots
$$

Example 19.5 Find all solutions to the equation

$$
\text { (i) } \quad \cos x=0, \quad \text { (ii) } \quad|\cos x|=1
$$

Solution (i). The main roots of this equation i.e. the zeros of the cosine function over its period from 0 to $360^{\circ}$ or in the arc measure in the range $0 \leq \alpha \leq 2 \pi$ are solutions

$$
x=\frac{\pi}{2}, \quad \text { or } \quad x=\frac{3 \pi}{2} .
$$

These solutions are marked on the graph $y=\cos x$.


We will get all the solution by adding the multiple of the period of the cosine function to the principal roots. Thus, all solutions have the form:

$$
x_{k}=\frac{\pi}{2}+2 k \pi, \quad \text { or } \quad x_{k}=\frac{3 \pi}{2}+2 k \pi,
$$

This means that all solutions are of the form:

$$
x_{k}=(2 k+1) \frac{\pi}{2}, \quad k=0, \pm 1, \pm 2, \ldots
$$

Solution (ii). Principal roots of the equation

$$
|\cos x|=1, \quad \text { or } \quad \cos x=1 \quad \text { or } \quad \cos x=-1
$$

are real numbers

$$
x=0, \quad \text { or } \quad x=\pi
$$

We will get all solutions by adding to the roots that are multiples of the period of the cosine function. Thus, all solutions have the form:

$$
x_{k}=2 k \pi, \quad \text { or } \quad x_{k}=\pi+2 k \pi=(2 k+1) \pi,
$$

This means that all solutions for even and odd $k$ are of the form:

$$
x_{k}=k \pi, \quad k=0, \pm, \pm 2, \ldots
$$

Note that the sine and cosine of the angles $\alpha_{k}=k \pi$ or $\alpha_{k}=(2 k+1) \frac{\pi}{2}$ can be written as a power of minus one:

$$
\sin (2 k+1) \frac{\pi}{2}=(-1)^{k}, \quad \cos k \pi=(-1)^{k}, \quad k=0, \pm 1, \pm 2, \ldots:
$$

Example 19.6 Find all solutions to the equation
(i) $\operatorname{tg} x=0, \quad$ (ii) $\quad|\operatorname{tg} x|=1$.
(iii) $\quad \operatorname{ctg} x=0, \quad(i v) \quad \operatorname{ctg} x \mid=1$.

Solution (i). Since the period of the tangent function is equal to $\pi$, the main root of the equation

$$
\operatorname{tg} x=0
$$

is equal to zero $x=0$.


We will get all solutions by adding a multiple of the period of the tangent function $\omega=\pi$ to the main root of the equation. Thus, all solutions have the form:

$$
x_{k}=k \pi, \quad k=0, \pm 1, \pm 2, \ldots ;
$$

Solution (ii). Within the period of the tangent function from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ there are two roots of the equation

$$
|\operatorname{tg}| x=1, \quad \text { or } \quad \operatorname{tg} x=1, \quad \operatorname{tg} x=-1
$$

These roots are $x_{1}=-\frac{\pi}{4}, x_{2}=\frac{\pi}{4}$.
We will get all solutions by adding to the main root a multiple of the period of the tangent function. Thus, all solutions have the form:

$$
x_{k}=-\frac{\pi}{4}+k \pi, \quad x_{k}=\frac{\pi}{4}+k \pi, \quad k=0, \pm 1, \pm 2, \ldots
$$

or written as one pattern

$$
x_{k}=(2 k+1) \frac{\pi}{4}, \quad k=0, \pm 1, \pm 2, \ldots
$$

Solution (iii). Note that $x=\frac{\pi}{2}$ is the root of the equation

$$
\operatorname{ctg} x=0
$$

This main root is marked on the function graph

$$
y=\operatorname{ctg} x
$$

in the range $(0, \pi)$


We can get all the solutions by adding to the main root a multiple of the period of the cotangent function. Thus, all solutions have the form:

$$
x_{k}=\frac{\pi}{2}+k \pi=(2 k+1) \frac{\pi}{2}, \quad k=0, \pm 1, \pm 2, \ldots
$$

Solution (iv). Principal roots of the equation

$$
|\operatorname{ctg} x|=1
$$

or equivalent equations

$$
\operatorname{ctg} x=1, \quad i \quad \operatorname{ctg} x=-1
$$

there are real numbers

$$
x_{1}=\frac{\pi}{4}, \quad x_{2}=\frac{3 \pi}{4}
$$

All solutions can be obtained by adding to the principal roots a multiple of the period of the cotangent function.
So, all solutions have the form:

$$
\begin{aligned}
& x_{k}=\frac{\pi}{4}+k \pi=(4 k+1) \frac{\pi}{4} \\
& \text { or } \\
& x_{k}=\frac{3 \pi}{4}+k \pi=(4 k+3) \frac{\pi}{4} \\
& \text { for } k=0, \pm 1, \pm 2, \ldots
\end{aligned}
$$

Checking. Putting into equation $x_{k}=\frac{\pi}{4}+k \pi$ or $x_{k}=\frac{3 \pi}{4}+k \pi$ we get

$$
\begin{array}{ll}
\operatorname{ctg} x_{k}=\operatorname{ctg}\left(\frac{\pi}{4}+k \pi\right)=\operatorname{ctg} \frac{\pi}{4}=1, & k=0, \pm 1, \pm 2, \pm 3, \ldots \\
\operatorname{ctg} x_{k}=\operatorname{ctg}\left(\frac{3 \pi}{4}+k \pi\right)=\operatorname{ctg} \frac{3 \pi}{4}=-1, & k=0, \pm 1, \pm 2, \pm 3, \ldots
\end{array}
$$

The values of trigonometric functions of selected angles are given in the table below.

| $\alpha$ | $\sin \alpha$ | $\cos \alpha$ | $\operatorname{tg} \alpha$ | $\operatorname{ctg} \alpha$ |
| :--- | :--- | :--- | :--- | :--- |
| $\alpha=0$ | 0 | 1 | 0 | $\infty$ |
| $\alpha=\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ | $\sqrt{3}$ |
| $\alpha=\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 | 1 |
| $\alpha=\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ | $\frac{\sqrt{3}}{3}$ |
| $\alpha=\frac{\pi}{2}$ | 1 | 0 | $\infty$ | 0 |
| $\alpha=\frac{3 \pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | -1 | -1 |
| $\alpha=\pi$ | 0 | -1 | 0 | $-\infty$ |
| $\alpha=\frac{5 \pi}{4}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | 1 | 1 |
| $\alpha=\frac{3 \pi}{2}$ | -1 | 0 | $\infty$ | 0 |
| $\alpha=\frac{7 \pi}{4}$ | $-\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | -1 | -1 |
| $\alpha=2 \pi$ | 0 | 1 | 0 | $\infty$ |

Example 19.7 Find all solutions to the equation

$$
\sin x-\cos x=0
$$

Solution. First, let us note that $D=R$ is the domain of the trigonometric expression.
From the table, we read the roots of the equation in the range $0 \leq x \leq 2 \pi$. Let us write the equation as follows

$$
\sin x=\cos x
$$

Then, we see that the sine is equal to cosine for the angles $x=\frac{\pi}{4}$ and $x=\frac{5 \pi}{4}$ which lie in the first or third quadrant of the trigonometric circle.
All solutions are obtained by adding $\omega=2 \pi$ to these solutions

$$
x_{k}=\frac{\pi}{4}+2 k \pi, \quad \text { or } \quad x_{k}=\frac{5 \pi}{4}+2 k \pi \quad k=0, \pm 1, \pm 2, \ldots ;
$$

We find the solution to this equation, in another way, by decomposing the trigonometric expression into factors. Namely, let's write the left side of the equation as

$$
\sin x-\sin \left(\frac{\pi}{2}-x\right)=0
$$

Applying the formula for the difference sins, we get the product

$$
\begin{aligned}
\sin x-\sin \left(\frac{\pi}{2}-x\right) & =2 \sin \frac{\left(\frac{\pi}{2}-x\right)-x}{2} \cos \left(\frac{\left.\frac{\pi}{2}-x\right)+x}{2}\right. \\
& =2 \cos \frac{\pi}{4} \sin \left(\frac{\pi}{4}-x\right) \\
& =\sqrt{2} \sin \left(\frac{\pi}{4}-x\right)=0
\end{aligned}
$$

Here the principal roots in the interval $[0.2 \pi]$ of the period of the sine function come from

$$
\frac{\pi}{4}-x=0, \quad \text { or } \quad \frac{\pi}{4}-x=\pi
$$

Let us add $\omega=2 \pi$ to the sine function, then we get all the solutions

$$
x_{k}=\frac{\pi}{4}+2 k \pi, \quad \text { or } \quad x_{k}=\frac{\pi}{4}+(2 k-1) \pi, \quad k=0, \pm 1, \pm 2, \ldots
$$

Example 19.8 Find all solutions to the equation

$$
\operatorname{tg} x+\operatorname{ctg} x=2
$$

Solution. From the values of the table of tangent and cotangent, we can see that the sum of the tangent and cotangent is equal 2 , when $\operatorname{tg} x=1$ and $\operatorname{ctg} x=1$ for $x=\frac{\pi}{4}$ or $x=\frac{5 \pi}{4}$. We will obtain all solutions by adding a multiple of their period to the main roots.
It means

$$
x_{k}=\frac{\pi}{4}+k \pi, \quad \text { or } \quad x_{k}=\frac{5 \pi}{4}+k \pi, \quad k=0, \pm 1 \pm 2, \ldots ;
$$

We get the same solutions in a different way. Namely, let's write this equation in its equivalent form

$$
\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}=2
$$

Note that the domain of the trigonometric expression in this equation is the set of real numbers $x \in R$ for which $\sin x \neq \mathrm{i} \cos x \neq 0$

$$
D=\left\{x \in R: \quad x \neq \frac{k \pi}{2},\right\}
$$

for integers $k=0, \pm 1, \pm 2, \ldots$;
We transform this equation using the trigonometric one and the double angle formula for sine

$$
\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}=\frac{\cos ^{2} x+\sin ^{2} x}{\sin x \cos x}=\frac{1}{\sin x \cos x}=2
$$

Hence, we get the equation

$$
2 \sin x \cos x=1, \quad \text { or } \quad \sin 2 x=1
$$

From the trigonometric function values in he table, we remember that $\sin 2 x=1$ for the root $x=\frac{\pi}{4}$ or $x=\frac{5 \pi}{4}$ in the trigonometric circle. Adding to the principal roots a multiple of the period $\omega=\pi$ of the function $\sin 2 x$, we get all solutions of this equation.

$$
x_{k}=\frac{\pi}{4}+k \pi, \quad \text { or } \quad x_{k}=\frac{5 \pi}{4}+k \pi \quad k=0, \pm 1, \pm 2, \ldots
$$

Note that the above roots of the equation are the same as in the first solution method and belong to the domain of equation.

Example 19.9 Solve the equation

$$
2 \sin ^{2} x-3 \sin x+1=0
$$

Solution. We solve this equation by substituting a new unknown $t=\sin x$ to get the quadratic equation

$$
2 t^{2}-3 t+1=0
$$

The discriminant of this equation $\Delta=(-3)^{2}-4 * 2 * 1=1$. Thus, solutions of the quadratic equation are

$$
t_{1}=\frac{3-1}{4}=\frac{1}{2}, \quad t_{2}=\frac{3+1}{4}=1 .
$$

Going back to the unknown $x$, we find all the solutions

$$
\begin{aligned}
& \sin x=\frac{1}{2}, \quad x_{k}=\frac{\pi}{6}+2 k \pi \\
& \text { or } \\
& \sin x=1, \quad x_{k}=\frac{\pi}{2}+2 k \pi
\end{aligned}
$$

for integer $k=0, \pm 1, \pm 2 \ldots$;
Question 19.19 Solve the equation

$$
2 \cos ^{2} x+\cos x-1=0
$$

One effective way to solve trigonometric equations is to factorize the trigonometric expression. Here is an example of such a method.

Example 19.10 Solve the equation

$$
\cos x+3 \cos 3 x+\cos 5 x=0
$$

Solution. Let's apply the formula for the sum of cosines

$$
\begin{aligned}
(\cos x+\cos 5 x)+\cos 3 x & =2 \cos \frac{x+5 x}{2} \cos \frac{x-2 x}{2}+\cos 3 x \\
& =2 \cos 3 x \cos (-2 x)+\cos 3 x \\
& =\cos 3 x(2 \cos 2 x+1)=0
\end{aligned}
$$

We have broken down the trigonometric expression into two factors that we equate to zero

$$
\cos 3 x=0, \quad \text { i } \quad 2 \cos 2 x+1=0, \quad \cos x-\frac{1}{2}
$$

Solving the above simple equations, we get the following series of solutions: When

$$
\cos 3 x=0
$$

we get the solutions

$$
\begin{aligned}
& 3 x=\frac{\pi}{2}+2 k \pi, \quad x_{k}=\frac{1}{6} \pi++\frac{2}{3} k \pi, \quad k=0, \pm 1, \pm 2, \ldots \\
& 3 x=\frac{3}{2} \pi+2 k \pi, \quad x_{k}=\frac{1}{2} \pi+\frac{2}{3} k \pi, \quad k=0, \pm 1, \pm 2, \ldots
\end{aligned}
$$

When $\cos 3 x=-\frac{1}{2}$, then we get the solutions

$$
\begin{aligned}
& 3 x=\frac{\pi}{3}+2 k \pi, \quad x_{k}=\frac{\pi}{9}+\frac{2}{3} k \pi, \quad k=0, \pm 1, \pm 2, \ldots \\
& 3 x=\frac{5}{3} \pi+2 k \pi, \quad x_{k}=\frac{5 \pi}{9}+\frac{2}{3} k \pi, \quad k=0, \pm 1, \pm 2, \ldots
\end{aligned}
$$

Example 19.11 Solve the equation

$$
\sin ^{2} x+2 \sin x-3=0
$$

Solution. Denote by $t=\sin x$. Then, we get the quadratic equation for the unknown $t$

$$
t^{2}+2 t-3=0
$$

whose solution is $t_{1}=-3$ and $t_{2}=1$. Since $-1 \leq \sin x \leq 1$, therefore $t=-3$ must be thrown. The value of $t=1$ remains. For that value

$$
\sin x=1, \quad \text { when } \quad x_{k}=\frac{\pi}{2}+2 k \pi, \quad k=0, \pm 1, \pm 2, \ldots
$$

### 19.5.1 Questions

Question 19.20 Solve the equation

$$
\text { (i) } \sin \frac{\pi x}{2}=0, \quad(i i) \quad \cos \frac{\pi x}{2}=0
$$

Question 19.21 Solve the equation

$$
\text { (i) } \operatorname{tg} \frac{\pi x}{2}=0, \quad \text { (ii) } \quad \operatorname{ctg} \frac{\pi x}{2}=0
$$

Question 19.22 Solve the equation

$$
\text { (i) } \quad \sin x+\cos x=0, \quad(i i) \quad \sin x=\cos x
$$

Question 19.23 Solve the equation

$$
\text { (i) } \quad \operatorname{tg} x-\operatorname{ctg} x=0 \quad \text { (ii) } \quad \operatorname{tg} x=\sin x
$$

Question 19.24 Solve the equation

$$
2 \cos ^{2} x-5 \cos x+2=0
$$

Question 19.25 Solve the equation

$$
2 \sin ^{3} x-\sin ^{2} x-2 \sin x+1=0
$$

Question 19.26 Solve the equation

$$
t^{3} x+3 t g^{2} x-3 t g x=1
$$

### 19.6 Trigonometric inequalities

Like trigonometric equations, we solve trigonometric inequalities using reduction formulas, sum formulas, and difference of trigonometric functions.

### 19.6.1 Fundamental inequalities for sine functions

Example 19.12 Solve the inequality in the range $[0.2 \pi]$.

$$
\text { (i) } \quad \sin x \leq \frac{1}{2}, \quad \text { (ii) } \quad \sin x>\frac{1}{2}
$$

Solution (i). The sine function gets $\sin x=\frac{1}{2}$ for angle $x=\frac{\pi}{6}$ in the first quadrant, or for $x=\frac{5 \pi}{6}$ in the second quadrant. So the inequality is true in the interval $[0.2 \pi]$ for

$$
0 \leq x \leq \frac{\pi}{6}
$$

or

$$
\frac{5 \pi}{6} \leq x \leq 2 \pi
$$

Let's see this solution in the graph of the sine function.


Note that the solution of $\sin x \leq \frac{1}{2}$ on the whole line of real numbers is those $x \in R$ points that belong to the segments

$$
x \in\left[a_{k}, b_{k}\right]=\left[\frac{\pi}{6}+2 k \pi, \frac{5 \pi}{6}+2 k \pi\right], \quad \text { when } \quad k=0, \pm 1, \pm 2, \pm 3 \ldots
$$

where $a_{k}=\frac{\pi}{6}+2 k \pi, \quad b_{k}=\frac{5 \pi}{6}+2 k \pi$.
(ii) Similarly, we find the solution to the opposite inequality $\sin x>\frac{1}{2}$


Solving the inequality $\sin x>\frac{1}{2}$, on the whole line of real numbers, are segments

$$
\begin{equation*}
\left[a_{k}, b_{k}\right]=\left[\frac{\pi}{6}+2 k \pi, \frac{5 \pi}{6}+2 k \pi\right], \quad k=0, \pm 1, \pm 2, \pm 3 \ldots \tag{19.7}
\end{equation*}
$$

starting at $a_{k}=\frac{\pi}{6}+2 k \pi$ and ending at $b_{k}=\frac{5 \pi}{6}+2 k \pi$.

### 19.6.2 Fundamental inequalities for cosine function

Example 19.13 Solve the inequality

$$
\begin{aligned}
& \text { (i) } \quad \cos x \leq \frac{1}{2} \\
& \text { (ii) } \cos x>\frac{1}{2}
\end{aligned}
$$

Solution (i). The cosine function becomes $\cos x=\frac{1}{2}$ for the angle $x=\frac{\pi}{3}$ in the first quart of the trigonometric circle or for the angle $x=\frac{5 \pi}{3}$ in the fourth quadrant of the trigonometric circle. So inequality

$$
\cos x \leq \frac{1}{2}
$$

is true in the range $[0,2 \pi]$ for the values of the angle $x \in\left[\frac{\pi}{3}, \frac{5 \pi}{3}\right]$.
Let's see this in the function graph $y=\cos x$


Solving the inequality $\cos x \leq \frac{1}{2}$, on the whole line of real numbers are segments

$$
\begin{equation*}
\left[a_{k}, b_{k}\right]=\left[\frac{\pi}{3}+2 k \pi, \frac{5 \pi}{3}+2 k \pi\right], \quad k=0, \pm 1, \pm 2, \pm 3 \ldots \tag{19.8}
\end{equation*}
$$

starting at $a_{k}=\frac{\pi}{2}+2 k \pi$ and ending at $b_{k}=\frac{5 \pi}{2}+2 k \pi$.
Solution (ii). The cosine function goes to $\cos x=\frac{1}{2}$ for kata $x=\frac{\pi}{3}$ in the first quadrant of the trigonometric circle or for kata $x=\frac{5 \pi}{3}$ in the fourth quadrant of the trigonometric circle. So inequality

$$
\cos x>\frac{1}{2}
$$

is true in the range $[0.2 \pi$ ] for the angle value

$$
x \in\left[0, \frac{\pi}{3}\right] \cup\left[\frac{5 \pi}{3}, 2 \pi\right]
$$



The solutions of the inequality $\cos x>\frac{1}{2}$, on the whole line of real numbers are the intervals

$$
\left[a_{k}, b_{k}\right]=\left[2 k \pi, \frac{\pi}{3}+2 k \pi\right], \quad k=0, \pm 1, \pm 2, \pm 3 \ldots ;
$$

at the beginning at $a_{k}=\frac{\pi}{3}+2 k \pi$ and ending at $b_{k}=\frac{5 \pi}{3}+2 k \pi$.
or intervals

$$
\left[c_{k}, d_{k}\right]=\left(\frac{5 \pi}{3}+2 k \pi, 2 k \pi\right], \quad k=0, \pm 1, \pm 2, \pm 3 \ldots ;
$$

at the beginning at $c_{k}=\frac{\pi}{3}+2 k \pi$ and the end at $d_{k}=\frac{5 \pi}{3}+2 k \pi$. ${ }^{6}$

### 19.6.3 Fundamental inequalities for tangent and cotangent

Tangent. As we know, the tangent function is specified for the argument

$$
x \neq(2 k+1) \frac{\pi}{2}, k=0, \pm 1, \pm 2, \ldots ;
$$

[^34]different from an odd multiple of a right angle $\frac{\pi}{2}$.


The tangent function is increasing. It means for greater values of the argument $x$ the values of the tangent function are greater, write
if $x_{1}<x_{2}$ are $\operatorname{tg} x_{1}<\operatorname{tg} x_{2}$. The tangent function is periodic with the limit $\omega=\pi$, we write

$$
\operatorname{tg} x=\operatorname{tg}(x+\pi) \quad \text { dla kazdego } x \neq \frac{(2 k+1) \pi}{2}, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots
$$

Let us consider the graph of the tangent function on the whole number line except the points

$$
x_{k} \neq(2 k+1) \frac{\pi}{2} \quad d l a \quad k=0 \pm 1, \pm 2, \pm 3, \ldots
$$

which the value of the tangent function is undefined.
Moving the graph of the function

$$
y=\operatorname{tg} x \quad \text { for } \quad x \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)
$$

by the period $\omega=\pi$ we will obtain plots of the tangent function in successive intervals of specificity

$$
x \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad x \in\left(\frac{\pi}{2}, \frac{3 \pi}{2}\right), \quad x \in\left(\frac{3 \pi}{2}, \frac{5 \pi}{2}\right), \quad x \in\left(\frac{5 \pi}{2}, \frac{7 \pi}{2}\right), \ldots .
$$

The values of the argument $x$ of the tangent function are marked below on the graph, for which the values of $\operatorname{tg} x$ are greater than or equal to one, i.e. the inequality is satisfied
$\operatorname{tg} x \geq 1$.


Example 19.14 .
(i) Find the solution of the inequality

$$
\operatorname{tg} x \geq 1
$$

in the open range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
(ii) Find all real solutions of inequalities

$$
\operatorname{tg} x \geq 1
$$

for $x \neq(2 k-1) \frac{\pi}{2}, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots$ :
Solution $(i)$. The function $y=\operatorname{tg} x$ is increasing in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
The value 1 is reached by the tangent function at $x=\frac{\pi}{4}$, means that $\operatorname{tg} \frac{\pi}{4}=1$.
So inequality

$$
\operatorname{tg} x \geq 1
$$

is true in the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for $x \in\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.
Solution (ii). From the graph of the function $y=\operatorname{tg} x$, we can see that in the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$
\operatorname{tg} x<1 \quad \text { for } \quad x \in\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)
$$

All the real solutions of the inequality $\operatorname{tg} x<1$ can be read from the graph. Namely, the value of the tangent function is less than one $\operatorname{tg} x<1$ for all real argument values belonging to the ranges

$$
\left(-\frac{\pi}{2}+k \pi, \frac{\pi}{4}+k \pi\right), \quad \text { dla } \quad k=0, p m 1, \pm 2, \ldots
$$

Question 19.27 Solve the inequality

$$
\begin{array}{ll}
\text { (i) } \operatorname{tg} x \geq \sqrt{3}, & \text { (ii) } \operatorname{tg} x<\sqrt{3} \\
\text { or all real values } & x \neq(2 k+1) \frac{\pi}{2}, \quad k=0, \pm 1, \pm 2, \ldots
\end{array}
$$

The cotangent function. As we know, the cotanges function is specified for the argument

$$
x \neq k \pi, \quad k=0, \pm 1, \pm 2, \ldots
$$

different from the semi-double angle multiple $\pi$.


The cotangent function is descending and periodic with the period $\omega=\pi$. That is for greater values of the argument $x$ the values of the cotanges function are smaller, write

$$
\text { if } x_{1}<x_{2} \text { are ctg } x_{1}>\operatorname{ctg} x_{2}
$$

The cotangent function is periodic with the period $\omega=\pi$, we write

$$
\operatorname{ctg} x=\operatorname{ctg}(x+\pi) \quad \text { for every } x \neq k \pi, \quad k=0, \pm 1, \pm 2, \ldots ;
$$

Example 19.15 .
(i) Find the solution of the inequality

$$
\operatorname{ctg} x \geq 1
$$

in the open interval $(0, \pi)$.
(ii) Find all real solutions of inequalities

$$
\operatorname{ctg} x \geq 1
$$

dla $x \neq k \pi, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots$ :
Solution $(i)$. The function $y=\operatorname{ctg} x$ decreases in the range $(0, \pi)$. The value of $\operatorname{ctg} x=1$ reaches for $x=\frac{\pi}{4}$.
Therefore

$$
\operatorname{ctg} x \geq 1 \quad \text { for } x \in\left(0, \frac{\pi}{4}\right)
$$

Solution $(i)$. The function $y=\operatorname{ctg} x$ decreases in the range $(0, \pi)$. The value of $\operatorname{ctg} x=1$ reaches for $x=\frac{\pi}{4}$.
Therefore

$$
\left(k \pi, \frac{\pi}{4}+k \pi\right), \quad \text { dla } \quad k=0, \pm 1, \pm 2, \ldots
$$

where the inequality of $\operatorname{ctg} x \geq 1$ is true
Question 19.28 .
(i) Find the solution of the inequality

$$
\operatorname{ctg} x \leq \frac{1}{\sqrt{3}}
$$

in the open interval $(0, \pi)$.
(ii) Find all real solutions of inequalities

$$
\operatorname{ctg} x>\frac{1}{\sqrt{3}}
$$

for $x \in(0, \pi)$.

## Question 19.29.

(i) Find the solution of the inequality

$$
\operatorname{ctg} x \geq \sqrt{3}
$$

in the open interval $(0, \pi)$.
(ii) Find all real solutions of inequalities

$$
\operatorname{ctg} x \geq \sqrt{3},
$$

for $x \neq k \pi, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots$ :
Question 19.30 .
(i) Find the solution of the inequality

$$
\operatorname{ctg} x \geq \operatorname{tg} x
$$

in the open interval $\left(0, \frac{\pi}{2}\right)$.
(ii) Find all real solutions of inequalities

$$
\operatorname{tg} x \geq c t g x
$$

for $x \neq \frac{k \pi}{2}, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots$ :
Question 19.31 .
Find the solution of the inequality

$$
t^{2} x-1 \geq 0
$$

in the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

### 19.7 Theorem of sines

Theorem 19.1 In any triangle, the ratio of the lengths of the sides to the sines of the angles lying on the opposite sides is constant and equal to the diameter of the circle circumscribed on the triangle. Thus the following equalities hold

$$
\begin{equation*}
\frac{a}{\sin \alpha}=\frac{b}{\sin \beta}=\frac{c}{\sin \gamma}=2 R \tag{19.9}
\end{equation*}
$$

There are some proofs for the theorem of sines. Here we give two proofs relating to a relationship between an inscribed circle and a center angle that are based on the same arc. Proof I. We consider a circle described on the triangle $\triangle A B C$ with radius $R$. From the vertex $A$ we lead the diameter of the circle to the intersection with the circle at point $D$. Note that the angles inscribed in the circle $\angle A B C=\beta$ and $\angle A D C=\delta$ are based on the
same arc $\widehat{A C}$. So they are equal to $\beta=\delta$. The triangle $\Delta A D C$ is right-angled because the angle $\angle D C A$ with vertex $C$ on the diameter is straight.


A circle encircled by a triangle $\triangle A B C$

From this right triangle $\triangle A D C$, we find the sine of the angle $\delta$. Namely

$$
\sin \delta=\frac{|A C|}{|A D|}=\frac{b}{2 R}, \quad \text { and } \quad \sin \beta=\frac{b}{2 R} \quad \text { dla } \beta=\delta
$$

Hence we calculate

$$
\begin{equation*}
\frac{b}{\sin \beta}=2 R \tag{19.10}
\end{equation*}
$$

Similarly, we prove the thesis for the angle $\alpha$ of the triangle $\triangle A B C$. We only change the position of the diameter, as in the picture
For the angle $\alpha$ we lead the diameter $C D$ from the vertex $C$ of the triangle $\triangle A B C$ to the intersection with the circle at the point $D$. Note that the angles inscribed in the circle $\angle C A D=\alpha$ and $\angle A D B=\delta$ are based on the same $\operatorname{arc} \widehat{B C}$. So they are equal to $\alpha=\delta$. The triangle $\triangle B C D$ is right-angled because the angle $\angle D B C$ with the vertex $B$ based on the diameter is straight.


A circle encircled by a triangle $\triangle A B C$
From this right triangle $\triangle B C D$, we find the sine of the angle $\delta$. Namely

$$
\sin \delta=\frac{|B C|}{|C D|}=\frac{a}{2 R} \quad \text { and } \quad \sin \alpha=\frac{a}{2 R} \quad \text { dla } \alpha=\delta
$$

Hence we calculate

$$
\begin{equation*}
\frac{a}{\sin \alpha}=2 R \tag{19.11}
\end{equation*}
$$

For the angle $\gamma$ we lead the diameter $B D$ from the vertex $B$ of the triangle $\Delta A B C$ to the intersection with the circle at the point $D$, as shown in the picture


A circle encircled by a triangle $\triangle A B C$
Note that the angles inscribed in the circle $\angle B C A=\gamma$ and $\angle B D A=\delta$ are based on the same arc $\widehat{A B}$. So they are equal to $\gamma=\delta$. The triangle $\Delta B D A$ is rectangular because the angle $\angle D A B$ based on the vertex $A$ based on the diameter is straight. From this rectangular triangle $\triangle B C D$, we find the sine of angle $\delta$. Namely

$$
\sin \delta=\frac{|A B|}{|B D|}=\frac{c}{2 R} \quad \text { and } \quad \sin \gamma=\frac{c}{2 R} \quad \text { for } \gamma=\delta
$$

Hence we calculate

$$
\begin{equation*}
\frac{c}{\sin \gamma}=2 R \tag{19.12}
\end{equation*}
$$

From the equality $(19.10,19.11,19.12)$ follows the thesis (19.9)

$$
\begin{equation*}
\frac{a}{\sin \alpha}=\frac{b}{\sin \beta}=\frac{c}{\sin \gamma}=2 R \tag{19.13}
\end{equation*}
$$

Proof II. Consider the triangle $\triangle A B C$. Let us lead the height $h$ from the vertex $C$ of the triangle $\triangle A B C$ to the base $A B$.

By making simple calculations

$$
\begin{aligned}
h & =a \sin \alpha \\
h & =b \sin \beta
\end{aligned}
$$

hence we get

$$
\begin{gathered}
a \sin \alpha=b \sin \beta \\
\frac{a}{\sin \alpha}=\frac{b}{\sin \beta}
\end{gathered}
$$

$$
\begin{aligned}
& h=b \sin \alpha \\
& h=b \sin \beta \\
& b \sin \alpha=a \sin \beta, \quad \text { we divide both sides by } \sin \alpha * \sin \beta
\end{aligned}
$$

So we get the expected equality

$$
\begin{equation*}
\frac{a}{\sin \alpha}=\frac{b}{\sin \beta} \tag{19.14}
\end{equation*}
$$

Equality is obtained in a similar way

$$
\frac{c}{\sin \gamma}=\frac{b}{\sin \beta}
$$

Namely, In the triangle $\triangle A B C$ we lead the height $h$ from the vertex $B$ to the base $A C$


By making simple calculations

$$
\begin{aligned}
& h=a \sin \gamma \\
& h=c \sin \alpha \\
& b \sin \alpha=a \sin \beta, \quad \text { we divide both sides by } \sin \alpha * \sin \gamma
\end{aligned}
$$

get the expected equality

$$
\begin{equation*}
\frac{a}{\sin \alpha}=\frac{c}{\sin \gamma} \tag{19.15}
\end{equation*}
$$

Comparing the equations (19.14) and (19.15) we get double the equality

$$
\begin{equation*}
\frac{a}{\sin \alpha}=\frac{b}{\sin \beta}=\frac{c}{\sin \gamma} \tag{19.16}
\end{equation*}
$$

It remains to prove that the ratio of the lengths of the sides of the triangle $\triangle A B C$ to the sines of the respective angles is constant and equal $2 R$, that is

$$
\frac{a}{\sin \alpha}=\frac{b}{\sin \beta}=\frac{c}{\sin \gamma}=2 R
$$

Consider once again the triangle $\triangle A B C$ and the circle inscribed on this triangle with radius $R$ and center at the point $O$. From the center of this triangle, let's take the radii to the
vertices $A, B, C$ and the height $h=|O D|$ of the triangle $\triangle B C O$, as in the picture


The circle is circumscribed by a triangle $\triangle A B C$
Note that the triangle $\triangle B C O$ is isosceles with the length of the legs equal to the radius of the circle $R$

$$
|O B|=|O C|=R
$$

So the point $D$, is the end of $h=|O D|$, divides the side $B C$ in half
From the theorem about the center angle and the angle inscribed in a circle based on the same arc widehat $B C$ it follows that the central angle is twice the inscribed angle,

$$
\text { angle inscribed in circle } \quad \angle B A C=\alpha \quad \text { and the angle with vedix in center } \quad \angle B O C=2 \alpha
$$

The height $h$ of an isosceles triangle $\triangle B C O$ divides the median angle $\angle C O B=2 \alpha$ by half. Thus, the angle $\angle C O B=\alpha$ in a right triangle $\triangle D C O$ with the hypotenuse $C O$ and the side $h=|O D| \mathrm{i}|D C|=\frac{a}{2}$.

From a right triangle $\triangle D C O$ we find

$$
\sin \alpha=\frac{a}{2 R}
$$

From where we get equality

$$
\begin{equation*}
\frac{a}{\sin \alpha}=2 R \tag{19.17}
\end{equation*}
$$

From the equality (19.16) and (19.17) follows the thesis theorems (19.9)

$$
\frac{a}{\sin \alpha}=\frac{b}{\sin \beta}=\frac{c}{\sin \gamma}=2 R
$$

We use the theorem of sines to find the sides and angles of a triangle on the basis of given

1. of two sides and an angle in one of them,
2. of a side and two angles adjacent to that side,

Example 19.16 Calculate the sides and angles of the triangle $\triangle A B C$, having the lengths of two sides $|A B|=c=4 i|B C|=a=2 k t \alpha=\frac{\pi}{6}$ opposite $[B C]$.

Solution. Select the data and the unknown sides and angles in the drawing

triangle $\triangle A B C$

From the theorem of sines we calculate the radius $R$ of the circle circumscribed by the triangle

$$
\frac{a}{\sin \alpha}=2 R, \quad \frac{2}{\sin \frac{\pi}{6}}=2 R, \quad \frac{2}{\frac{1}{2}}=2 R, \quad R=2
$$

Then also from the theorem of sines we calculate the sine of the angle $\gamma$ which lies opposite the side $|A B|=c=2$

$$
\frac{c}{\sin \gamma}=2 R, \quad \sin \gamma=\frac{c}{2 R}=\frac{2}{4}=\frac{1}{2}
$$

From where we find the angle $\gamma=\frac{\pi}{6}$ and angle $\beta$ from the sum of the angles in the triangle

$$
\alpha+\beta+\gamma=\pi, \quad \beta=\pi-\alpha-\gamma=\pi-\frac{\pi}{6}-\frac{\pi}{6}=\frac{2 \pi}{3}
$$

The remaining side $|A C|=b$ is calculated from the theorem of sines

$$
\frac{b}{\sin \beta}=2 R, \quad b=2 R \sin \beta=2 * 2 * \sin \frac{2 \pi}{3}=4 \frac{\sqrt{3}}{2}=2 \sqrt{3}
$$

### 19.8 Theorem of cosines

Like the theorem of sines, we use the theorem of cosines to calculate the sides and angles of any triangles.

Theorem 19.2 In any triangle with the sides and angles marked in the drawing below,


Triangle $\triangle A B C$

There are the following relations
(i) $a^{2}=b^{2}+c^{2}-2 b c \cos \alpha$
(ii) $b^{2}=a^{2}+c^{2}-2 a c \cos \beta$
(iii) $c^{2}=a^{2}+b^{2}-2 a b \cos \gamma$

Proof. We will prove the first of the above equations. Note that in the case of a right triangle, when $\alpha=\frac{\pi}{2}$ formula (i) is true, because then $\cos \alpha=0$ and uses the Pythagorean theorem. For $\alpha<\frac{\pi}{2}$. Point $\mathbf{D}$, saucer of $\mathbf{h}$, divides the side of $[A B]$ into two parts $|A D|$ and $D B \mid$, where ${ }^{7}$

$$
|A D|=b \cos \alpha, \quad i \quad|D B|=a \cos \beta
$$

Applying the Pythagorean theorem to the triangles $\triangle A D C$ and $\triangle D B C$, we compute

$$
\begin{aligned}
& h^{2}=|A C|^{2}-|A D|^{2} \\
& \text { and } \\
& h^{2}=|B C|^{2}-|B D|^{2}
\end{aligned}
$$

Comparing the right sides of the above equations, we get the following equality

$$
\begin{equation*}
|A C|^{2}-|A D|^{2}=|B C|^{2}-|D B|^{2} \tag{19.18}
\end{equation*}
$$

Then substituting for equality (19.18)

$$
\begin{array}{lll}
a=|B C|, & b=|A C|, & c=|A B|, \\
|A D|=b \cos \alpha, & |D B|=c-|A D|, & |D B|=c-b \cos \alpha
\end{array}
$$

we will obtain the relation $(i)$, in a simple form, between the sides and angles of the triangle $\triangle A B C$

$$
\begin{equation*}
\underbrace{b^{2}-(b \cos \alpha)^{2}}_{|A C|^{2}-|A D|^{2}}=\underbrace{a^{2}-(c-b \cos \alpha)^{2}}_{|B C|^{2}-|D B|^{2}} \tag{19.19}
\end{equation*}
$$

Simplifying the above algebraic expressions

$$
b^{2}-(b \cos \alpha)^{2}=a^{2}-\left(c^{2}-2 b c \cos \alpha+b^{2} \cos ^{2} \alpha\right)
$$

we get the first thesis $(i)$ the theorem of cosines

$$
a^{2}=b^{2}+c^{2}-2 b c \cos \alpha
$$

We prove the other formulas (ii) and (iii) similarly. Namely, in order to derive the formula (ii), one should plot the height $h$ from the vertex $A$ to the side of $B C$. On the other hand, for the proof of thesis (iii), one should plot the height $h$ from the vertex $B$ of the triangle $\triangle A B C$ aside $A C$.

[^35]Example 19.17 In the triangle $\triangle A B C$, the lengths of the side $|A B|=3$ and the side $|A C|=3$ are given, and the value of the angle between them $\alpha=\frac{\pi}{3}$, as in the picture, calculate the side length $\mathbf{a}$ and the value of angles $\beta, \gamma$.

triangle $\triangle A B C$

Solution. Calculate from the theorem of cosines

$$
\begin{aligned}
a^{2}=b^{2}+c^{2}-2 b c \cos \alpha & =3^{2}+3^{2}-2 * 3 * 3 * \frac{1}{2} \\
& =9+9-9=9
\end{aligned}
$$

Hence side length $a=\sqrt{9}=3$.
Given the sides of the triangle, $a, b, c$ we calculate the cosines of the angles $\beta$ and $\gamma$.

$$
\begin{aligned}
& \cos \beta=\frac{a^{2}+c^{2}-b^{2}}{2 a c}=\frac{3^{2}+3^{2}-3^{2}}{2 * 3 * 3}=\frac{1}{2} \\
& \cos \gamma=\frac{a^{2}+b^{2}-c^{2}}{2 a b}=\frac{3^{2}+3^{2}-3^{2}}{2 * 3 * 3}=-\frac{1}{2}
\end{aligned}
$$

If $\cos \beta=\frac{1}{2} \mathrm{i} \cos \gamma=\frac{1}{2}$ then tha angle $\beta=\frac{\pi}{3}$ i ka̧t $\gamma=\frac{\pi}{3}$.

### 19.8.1 Questions

Question 19.32 Calculate the sides and angles of $\triangle A B C$ having the lengths of two sides

$$
|A B|=5, \quad|A C|=5 \quad i \quad \text { kat } \angle A B C=30^{\circ}
$$

Question 19.33 Find the sides of the triangle $\triangle A B C$ having the side length $|B C|=25$ and angles

$$
\angle C A B=30^{\circ}, \quad \angle A B C=60^{\circ}
$$

Question 19.34 Find the angles of the triangle $\triangle A B C$ having the side lengths

$$
|A B|=15 m, \quad|B C|=30 m, \quad|A C|=45 m
$$

### 19.9 Cyclometric functions

Cyclometric functions or circular functions

$$
\begin{array}{lll}
y=\arcsin x, & x \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) & y=\arccos x,
\end{array} \quad x \in[0, \pi)
$$

are inverse functions to trigonometric functions in intervals, where the trigonometric functions are monotonic i.e. increasing or decreasing.
For example, the inverse function to $x=\sin y$ is $; y=\arcsin x \quad$ specified in the range, where the function sine is monotonic i.e. when $x \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. In general

$$
\text { when } x \in\left(-\frac{\pi}{2}+2 k \pi, \frac{\pi}{2}+2 k \pi\right), \quad k=0, \pm 1 \pm 2, \pm 3, \ldots
$$

The inverse function to $x=\cos y \quad$ is $y=\arccos x$, which is specified in the the open interval $\alpha \in(0, \pi)$ or in general

$$
\text { when } \quad x \in\left(-\frac{\pi}{2}+2 k \pi, \frac{\pi}{2}+2 * k \pi\right), \quad k=0, \pm 1, \pm 2, \pm 3, \ldots
$$

Similarly, the inverse functions of $x=\operatorname{tg} y \quad$ and $\quad x=\operatorname{ctg} y$ are $y=\operatorname{arctg} x \quad$ and $y=\operatorname{arcctg} x$, specified in open intervals tangent for

$$
x \in\left(-\frac{\pi}{2}+k \pi, \frac{\pi}{2}+k \pi\right), \quad k=0, \pm 1, \pm 2, \pm 3, \ldots
$$

and cotangens for

$$
\alpha \in(k \pi,(k+1) \pi), \quad k=0, \pm 1, \pm 2, \pm 3, \ldots
$$

### 19.9.1 Arcsine

The function $y=\sin x$ is increasing in a closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The set of values of this function is the interval $[-1.1]$. Thus the inverse function

$$
x=\arcsin y
$$

to the function $y=\sin x$ exists and is defined in the closed interval $[-1,1]$. That is, the domain of the inverse function $x=\arcsin y$ to the function $y=\sin x$ is the set of function values $y=\sin x$.

So for the inverse function $x=\arcsin y$ the independent variable is $y \in[-1,1]$, and the dependent variable is

$$
x \in\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]
$$

On the graph the arcsine function, we change the roles of $x, y$ as below

$$
y=\arcsin x, \quad y \in\left[\frac{-\pi}{2}, \frac{\pi}{2}\right], \quad \text { when } \quad x \in[-1,1]
$$



The values of the function $y=\arcsin x \quad$ for the selected values of $x$ are given in the array

| $x^{o}$ | radian | $y=\sin x$ | $x=\arcsin y$ |
| :---: | :---: | :---: | :---: |
| $-90^{\circ}$ | $-\frac{\pi}{2}$ | -1 | $-\frac{\pi}{2}$ |
| $-60^{\circ}$ | $-\frac{\pi}{3}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\pi}{3}$ |
| $-45^{\circ}$ | $-\frac{\pi}{4}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\pi}{4}$ |
| $-30^{\circ}$ | $-\frac{\pi}{6}$ | $-\frac{1}{2}$ | $-\frac{\pi}{6}$ |
| $0^{\circ}$ | 0 | 0 | 0 |
| $30^{\circ}$ | $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\pi}{6}$ |
| $45^{\circ}$ | $\frac{\pi}{4}$ | 0 | $-\frac{\pi}{4}$ |
| $60^{\circ}$ | $\frac{\pi}{3}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\pi}{3}$ |
| $90^{\circ}$ | $\frac{\pi}{2}$ | 1 | $\frac{\pi}{2}$ |

Note that the following identities hold
(i) $\quad \arcsin (\sin x) \equiv x, \quad$ dla $\quad-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
(ii) $\sin (\arcsin x) \equiv x \quad$ dla $\quad-1 \leq x \leq 1$

Indeed, let

$$
y=\sin x \quad \text { for } \quad x \in\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]
$$

Then the function $\sin x$ is ascending and the function inverse $x=\arcsin y$ exists and is defined for $y \in[-1,1]$.
Substituting (i) $y=\sin x$ to the left, we get identity (i).
Likewise, let

$$
y=\arcsin x \quad \text { for } \quad x \in[-1,1] .
$$

Then the function $\arcsin x$, is incremental and the function inverse $x=\sin y$ exists and is defined for $y \in[-1,1]$.
Substituting $y=\arcsin x$, into equality $x=\sin y$, we get the identity (ii).

### 19.9.2 Arccosine

The descending function $y=\cos x$ in the interval $[0, \pi]$. The set of values of the cosine function is the interval $[-1,1]$. The inverse function $x=\arccos y$ to the function $y=$ $\cos x$, exists and is defined in the interval $[-1.1]$. Thus the domain of the inverse function

$$
x=\arccos y
$$

to the function

$$
y=\cos x
$$

is the set of values of the function $y=\cos x$. On the other hand, the set of values of the function $x=\arccos y$ is the interval $[0, \pi]$.
Below on the graph in the arccosine function we change the roles of the variables $x, y$ assuming


| $x^{\circ}$ | radian | $y=\cos x$ | $x=\arccos y$ |
| :--- | :--- | :--- | :--- |
| 0 | 0 | 1 | 0 |
| $30^{\circ}$ | $\frac{\pi}{6}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\pi}{6}$ |
| $45^{\circ}$ | $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\pi}{4}$ |
| $60^{\circ}$ | $\frac{\pi}{3}$ | $\frac{1}{2}$ | $\frac{\pi}{3}$ |
| $90^{\circ}$ | $\frac{\pi}{2}$ | 0 | $\frac{\pi}{2}$ |
| $135^{\circ}$ | $\frac{3 \pi}{4}$ | $-\frac{\sqrt{3}}{2}$ | $\frac{3 \pi}{4}$ |
| $150^{\circ}$ | $\frac{5 \pi}{6}$ | $-\frac{\sqrt{3}}{2}$ | $\frac{5 \pi}{6}$ |
| $180^{\circ}$ | $\pi$ | -1 | $\pi$ |

There is a simple relation between arcosine and arccosine, namely

$$
\begin{equation*}
\arcsin x+\arccos x=\frac{\pi}{2} \tag{19.20}
\end{equation*}
$$

Indeed, we notice that the inequality

$$
0 \leq \frac{\pi}{2}-\arcsin x \leq \pi i s t r u e
$$

From the inequality

$$
-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}
$$

and with equality

$$
\cos \left(\frac{\pi}{2}-\arccos x\right)=\sin (\arcsin x)
$$

follows the identity (19.20).

### 19.9.3 Arctangent

The function tangent $y=\operatorname{tg} x$ is periodic with the period $\omega=\pi$. This function is specified for an argument different than an odd multiple of a right angle $\frac{\pi}{2}$.

$$
x \neq(2 k+1) \frac{1}{2}, k=0, \pm 1, \pm 2, \ldots
$$

The interval of the values of tangent is the set of real numbers $R=(-\infty, \infty)$.
The function $y=\operatorname{tg} x$ increases from $-\infty$ to $\infty$ in the open interval

$$
x \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)
$$

Therefore, there is an inverse function

$$
x=\operatorname{arctg} y
$$

to the function $y=\operatorname{tg} \mathrm{x}$ in the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Without changing the properties of the arctangent function we can replace $x$ by $y$ as below

$$
y=\arctan x
$$

Values of the arctangent function belong to the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we write

$$
-\frac{\pi}{2}<\operatorname{arctg} x<\frac{\pi}{2}, \quad-\infty<x<\infty
$$

Let us consider the graph of the function tangent. Note that the plot of arctangent function to function tangent is obtained by rotating the plot of tangent counterclockwise looking at
the plot from the reverse side.


From the plot below we see the interval $-\infty<x<\infty$ for which the arctangent is specified and the interval $-\frac{\pi}{2}<y<\frac{\pi}{2}$ of values of the function

$$
y=\operatorname{arctg} x \text { for } x \in(-\infty, \infty)
$$



Function arctag $x$ has two asymptots $L_{1} i L_{1}$ parallel to exis $x$

### 19.9.4 Arcotangent

The function $y=\operatorname{ctg} x$ is decreasing over the open interval $(0, \pi)$ and its set values are all real numbers $-\infty<y<\infty$. So the inverse function $x=\operatorname{arcctg} y$ exists and is specified for all reales $-\infty<y<\infty$. However, its set of values varies from 0 to $\pi$, that is

$$
0<\operatorname{arcctg} y<\pi, \quad-\infty<y<\infty
$$

As in the case of the arcu tangent function, without changing the properties of the arcus cotangent function, we replace the $x, y$ variables with places, writing

$$
y=\operatorname{arcctg} x, \quad \text { for }-\infty<x<\infty
$$

Consider the graph of the cotangent function again. Note that the inverse cotangent plot is obtained by rotating the cotangent plot counterclockwise looking at the plot from the reverse side.


Let us use the plot above of the cotangent function to draw a graph of the inverse arccotangent function

$$
y=\operatorname{arcctg} x, \quad \text { for } \quad-\infty<x<\infty
$$



Function arcctg $x$ has two asymptots $L_{1}$ to jest os $x i L_{2}$ rownolegla do osi $x$
The following relationship between the arctangent and arcus cotangent holds

$$
\begin{equation*}
\operatorname{arctg} x+\operatorname{arcctg} x=\frac{\pi}{2} \tag{19.21}
\end{equation*}
$$

Indeed, we notice that the angle given by arccontangent satisfies the inequality

$$
0 \leq \frac{\pi}{2}-\operatorname{arctg} x \leq \pi
$$

Because $-\frac{\pi}{2} \leq \operatorname{arctg} x \leq \frac{\pi}{2}$ thus we have the identity

$$
\operatorname{ctg}\left(\frac{\pi}{2}-\operatorname{arctg} x\right)=\operatorname{arctg} x
$$

Hence shows the identity (19.21).

### 19.10 Questions

Question 19.35 Calculate the value of the expression

$$
\arcsin \frac{1}{2}+\arcsin \frac{\sqrt{2}}{2}
$$

Question 19.36 Calculate the value of the expression

$$
\arccos \frac{1}{2}+\operatorname{arcscos} \frac{\sqrt{2}}{2}
$$

Question 19.37 Calculate the value of the expression

$$
\operatorname{arctg} \frac{1}{\sqrt{2}}+\operatorname{arctg} \sqrt{2}
$$

Question 19.38 Provide a graph of the function

$$
f(x)=\sin (\arcsin x)
$$

for $x \in[-1,1]$.
Question 19.39 Provide a graph of the function

$$
f(x)=\cos (\arcsin x)
$$

for argumentu $x \in[-1,1]$.
Question 19.40 Solve the equation

$$
\arcsin x-\arccos x=0
$$

Question 19.41 Solve the equation

$$
2 \arcsin x-\arccos x=\pi
$$

Question 19.42 Solve the equation

$$
\operatorname{arctg} x-\operatorname{arcctg} x=0
$$

Question 19.43 Solve the equation

$$
3 \operatorname{arctg} x-2 \operatorname{arcctg} x=\pi
$$

## Chapter 20

## Combinatorics

Combinatorics includes terms such as factorial of a natural number $n$, permutations, variations without repetition and variations with repetition, combinations.
A description of these concepts with examples and exercises is preseted below.

### 20.0.1 Factorial of a natural number $n$ !

The product of consecutive natural numbers up to and including $n$ is called the factorial of $n$ and denoted by the symbol $n$ !. So we have

$$
n!=1 * 2 * 3 * \cdots *(n-1) * n
$$

It is acceepted that $0!=1$
Let us list a few factorials of consecutive natural numbers

$$
\begin{aligned}
& 0!=1 \\
& 1!=1 \\
& 2!=1 * 2=2 \\
& 3!=1 * 2 * 3=6 \\
& 4!=1 * 2 * 3 * 4=24 \\
& 5!=1 * 2 * 3 * 4 * 5=120 \\
& 6!=1 * 2 * 3 * 4 * 5 * 6=720 \\
& 7!=1 * 2 * 3 * 4 * 5 * 6 * 7=5040 \\
& \cdots \cdots \cdots \cdots \cdots \cdots \ldots \ldots \ldots \ldots \\
& \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
& (n-1)!=1 * 2 * 3 * 4 * 5 * \cdots *(n-1) \\
& n!=1 * 2 * 3 * 4 * 5 * 6 * 7 * \cdots *(n-1) * n
\end{aligned}
$$

### 20.0.2 Examples

We explain the calculation of the factorials on the following examples
Example 20.1 Calculate the value of the fraction

$$
\frac{5!* 7!}{4!* 6!}
$$

## Solution

We can easily simplify this fraction by writing

$$
\begin{aligned}
& 5!=4!* 5, \quad 7!=6!* 7 \\
& \frac{5!* 7!}{4!* 6!}=\frac{4!* 5 * 6!* 7}{4!* 6!}=5 * 7=42
\end{aligned}
$$

Example 20.2 Simplify the fraction

$$
\frac{n!}{(n-1)}!
$$

## Solution

We can easily simplify this fraction by writing

$$
\begin{aligned}
(n-1)! & =1 * 2 * 3 * 4 * 5 * 6 * 7 * \cdots *(n-1) \\
n! & =1 * 2 * 3 * 4 * 5 * 6 * 7 * \cdots *(n-1) * n \\
\frac{n!}{(n-1)!} & =\frac{1 * 2 * 3 * 4 * 5 * 6 * 7 * \cdots *(n-1) * n}{1 * 2 * 3 * 4 * 5 * 6 * 7 * \cdots *(n-1)}=n
\end{aligned}
$$

Zasanie 20.1 Calculate the value of the fraction

$$
\frac{3!* 5!* 7!* 9!}{2!* 4!* 6!* 8!}
$$

Zasanie 20.2 Simplify the fraction

$$
\frac{2 n!}{(2 n-3)!}
$$

### 20.0.3 Permutations

A permutation of elements of a set is to put them in a certain order. Two permutations that consist of the same elements are different, if they differ in the order of the elements.

For example:
Permutations of digits of a two-digit number 23 consist of the same digits 2 and 3 but create two different permutations

$$
23 \quad i \quad 32 \quad \text { number ofpermutations } \quad 2!=2
$$

Note that there are no other permutations of 2 and 3 .
Similarly, let's list all permutations of digits of the three digit number 257

| 257 | 275 |
| :--- | :--- |
| 527 | 572 |
| 725 | 752 |$\quad$ number pf permutations $\quad 3!=6$

Example 20.3 List all permutations of the two-element set $a, b$

$$
a b \quad b a \quad \text { ilosc permutacji } \quad 2!=2
$$

Note that there are no other permutations of the letters $a$ and $b$.

Similarly, let's list all permutations of the three-element set $a, b, c$

$$
\begin{array}{ll}
a b c & a c b \\
b a c & b c a \\
c a b & c b a
\end{array} \quad \text { number of permutations } \quad 3!=6
$$

Generally, the number of permutations of an n-element set is $n$ !
Zasanie 20.3 List all the digit permutations of a three-digit number 391
Zasanie 20.4 List all permutations of the elements of the set four-item $A B C D$

### 20.1 Variations

A set with k-elements selected from an n-element set $(n \geq k)$ is called variation $k$ elements selected from the n-element set. We distinguish between variations with repetitions and variations without repetitions
If in a k-elements variation occure the same elements then this variation is called with repetations, otherewise variation without repetitions.
In variations with repetitions or without repetions is important order of their elements. It means that two variations are different if they consist of the same elements but differ in the order of their elements.

### 20.1.1 Variations with repetitions.

The concept of variations without repetitions or with repetitions is well illustrated by a game, to chose k elements from an n -element set that contains only different elements.

Namely, we create variations with repetitions in such a way that we throw the drawn element back into the urn before choosing the next element. We contnue the game until we select k -elements. In this way, we get a sequence of k-elements i.e. variantion with repetition.

Similarly, we create k-element variations without repetition, with the difference that we do not throw the drawn element back into the urn before drawing the next elements. This way we get a k-element variation in which all elements are different, i.e. there are no repeated elements.

The number of possible k-element variations with repetitions formed from the set of nelements is calculated from the formula

$$
V_{n}^{k}=n^{k}
$$

### 20.1.2 Examples

The concept of variations with repetitions we illustrate and explaine in the following examples
Example 20.4 List all two-digit numbers made up of the $\{1,2\}$ set of digits.

## Solution:

In this example, two-digit numbers are 2 -element variations with repetitions from a 2 element set. We find easily

Answer. The number of two-digit numbers made up of digits 1 and 2 is the number of variations with repetitions $V_{2}^{2}=2^{2}=4$
Example 20.5 List all two-digit numbers made up of the $\{1,2,3\}$ set of digits.

## Solution

In this example, two-digit numbers are 2-element variations with repetitions from a 3 element set. We find these numbers easily

| 11 | 12 | 13 |
| :--- | :--- | :--- |
| 21 | 22 | 23 |
| 31 | 32 | 33 |

Answer. The number of two-digit numbers formed digit 1, 2, 3 is the number of variations with repetition $V_{3}^{2}=3^{2}=9$

Example 20.6 List all three-digit numbers made up of a set of digits $\{1,2,3\}$.

## Solution.

In this example, three-digit numbers are 3-element variations with repetitions from a 3element set. We can easily find three-digit numbers

| 111 | 122 | 113 |
| :--- | :--- | :--- |
| 121 | 122 | 123 |
| 131 | 132 | 133 |
| 211 | 212 | 213 |
| 221 | 122 | 123 |
| 231 | 132 | 233 |
| 311 | 312 | 313 |
| 321 | 322 | 323 |
| 331 | 332 | 333 |

Answer. The number of three-digit numbers formed digit 1, 2,3 is the number of variations with repetition $V_{3}^{3}=3^{3}=27$
Zasanie 20.5 List all 2-element variations with repetitions created from a 3-element set $\{a, b, c\}$.

Zasanie 20.6 List all 3-element variations with repetitions created from a 3-element set $\{a, b, c\}$.

Zasanie 20.7 List all two-digit numbers made up of a set of digits $\{2,5,7,9\}$.

### 20.1.3 Variations without repetition

A k-element variation without repetition is a sequence of different elements selected from n-elements set $(1 \leq k \leq n)$.

The number of all k-element variations without repetition selected from the n-element set is given by the formula:

$$
W_{n}^{k}=\frac{n!}{(n-k)!}=(n-k+1) *(n-k+2) * \cdots *(n-1) * n
$$

or by writing the product in the reverse order of its factors, we have the formula

$$
W_{n}^{k}=\frac{n!}{(n-k)!}=n *(n-1) * \cdots *(n-k) *(n-k+1)
$$

### 20.1.4 Examples

The concept of variations without repetitions and the calculation of the number of k-element variations without repetitions selected from the n-element set are illustrated and explained on the following examples

Example 20.7 List all two-digit numbers with different digits made up of the set of digits $\{1,2\}$.

## Solution.

In this example, two-digit numbers are 2-element variations without repetition, selected from a 2 -element set. We find these numbers easily
$12 \quad 21$
Answer. The number of two-digit numbers with different digits made up of the numbers 1 and 2 is the number of variations without repetition. In this case it is equal to the number of permutations $W_{2}^{2}=2!=2$

Example 20.8 List all two-digit numbers with different digits made up of the set of digits $\{1,2,3\}$.

## Solution.

In this example, the two-digit numbers are 2-element variations without repetition selected from a 3 -element set. We find these numbers easily

$$
\begin{array}{ll}
12 & 13 \\
21 & 23 \\
31 & 32
\end{array}
$$

Answer. The number of two-digit numbers with different digits created digits 1, 2, 3 is the number of variations without repetition

$$
W_{3}^{2}=\frac{3!}{(3-2)!}=\frac{6}{1}=6
$$

Example 20.9 List all three-digit numbers with different digits made up of the set of digits $\{1,2,3\}$.

## Solution.

In this example, three-digit numbers are 3-element variations without repetition, selected from a set that also includes 3 elements. We easily find these three-digit numbers with different digits

$$
\begin{array}{ll}
123 & 132 \\
213 & 231 \\
312 & 321
\end{array}
$$

Answer. The number of different-digit three-digit numbers made up of the digits 1,2 , 3 is the number of variations without repetition. In this example, this is the number of permutations $W_{3}^{3}=3!=6$

Zasanie 20.8 List all 2-element variations without repetition created from the 3-element set $\{a, b, c\}$.

Zasanie 20.9 List all 3-element variations without repetition created from a 3-element set $\{a, b, c\}$.

Zasanie 20.10 List all two-digit numbers with different digits made up of the set of digits $\{2,5,7,9\}$.

Zasanie 20.11 List all 2-element variations without repetition selected from thea 4-element $\operatorname{set}\{a, b, c, d\}$.

### 20.1.5 Combinations

A combination of a k-elements selected from a set of $n$-elements is a k-element subset of the set of n-elements. Thus, in a combination, the order of the elements is not important. Therefore two combinations are different, if they are different at least by one element.

We calculateThe number of combinations from the following formul

$$
C_{n}^{k}=\frac{n!}{k!(n-k)!}
$$

or using the Newton symbol we write

$$
\binom{n}{k}=\frac{n!}{k!(n-k)!}
$$

Thus, the number of combinations consisting of $k$ elements selected from a n-element set is equal to the number of subsets of the set.

### 20.1.6 Examples

The concept of combinations and the calculation of number of k-elements combinations selected from the n-element set, we explain on the following examples
Example 20.10 How many chess pairs can be made in a class of 20 students so that each student plays only once with each selected student?

## Solution.

The number of pairs selected of 20 students is equal to the number of 2-element combinations from a 20 -element set,
Each student may select a chess partner from $20-1=19$ others So the number of differe.nt pairs is equal

$$
\frac{19 * 20}{2}=190
$$

The number of 2 -element combinations from the 20 -element set is also calculated from the formula

$$
C_{20}^{2}=\frac{20!}{2!(20-2)!}=\frac{19 * 20}{2}=190
$$

Example 20.11 There are 15 students in a class. How many ways can you choose
(i) three representatives
(ii) four representatives

Solution $(i)$ The nummber of representative consisting of three students is equal to thenumber of combinations consisting three chosen of 15 .

$$
C_{15}^{3}=\frac{15!}{3!(15-3)!}=\frac{13 * 14 * 15}{6}=13 * 7 * 5=455
$$

Answer. The number of possible student representatives in groups of 3 is 455 threes The solution (ii) is similar

Example 20.12 How many possible outcomes are there in the number game game if we choose six numbers outoff forty nine numbers?

## Solution.

The number of possible outcomes is equal to the number of 6 element combinations selected from the 49 elements of the set.
So, we calculate of te using the formula

$$
C_{49}^{6}=\frac{49!}{6!(49-6)!}=\frac{43 * 44 * 45 * 46 * 47 * 48 * 49}{1 * 2 * 3 * 4 * 5 * 6}=13983816
$$

Answer. In the number game possible quantity of outcomes $13,983,816$.
Example 20.13 Six points are marked on the circle $p_{1}, p_{2}, p_{3}, p_{4}, p_{5}, p_{6}$


Polygons with vertices on the circle
How many different polygons can you draw?
(a) triangles
(b) quadrilaterals
(c) pentagons
(d) hexagons
with vertices at points of the circle $p_{1}, p_{2}, p_{3}, p_{4}, p_{5}, p_{6}$

## Solution.

Two polygons are different if they differ at least one vertex. Likewise, two combinations are different if they differ at least one element.
So, the number of triangles is equal to the number of 3 -elements combinations selected from the 6 -elements set. The number of possible triangles equals

$$
C_{6}^{3}=\frac{6!}{3!(6-3)!}=\frac{1 * 2 * 3 * 4 * 5 * 6}{1 * 2 * 3 * 1 * 2 * 3}=\frac{4 * 5 * 6}{1 * 2 * 3}=4 * 5=20
$$

Similarly, the number of quadrilaterals is equal to the number of 4-element combinations selected from the 6 -element set. The number of possible We calculate quadrilaterals with vertices on the circle using the formula

$$
C_{6}^{4}=\frac{6!}{4!(6-4)!}=\frac{1 * 2 * 3 * 4 * 5 * 6}{1 * 2 * * 3 * 4 * 1 * 2}=\frac{5 * 6}{1 * 2}=15
$$

The number of pentagons is equal to the number of 5 -element combinations selected from the 6 -element set. The number of possible We calculate pentagons with vertices on the circle using the formula

$$
C_{6}^{5}=\frac{6!}{5!(6-5)!}=\frac{1 * 2 * 3 * 4 * 5 * 6}{1 * 2 * 3 * 4 * 5 * 1}=\frac{6}{1}=6 .
$$

The number of hexagons is equal to the number of 6 -element combinations selected from the 6 -element set. The number of possible we calculate hexagons with vertices on a circle from the formula

$$
C_{6}^{6}=\frac{6!}{6!(6-6)!}=\frac{1 * 2 * 3 * 4 * 5 * 6}{1 * 2 * 3 * 4 * 5 * 6 * 0!}=1, \quad g d y z \quad 0!=1
$$

### 20.2 Questions

Question 20.1 Simplify the arithmetic exprssion

$$
\frac{3!* 5!(5-2)!}{9!-7!)}
$$

Question 20.2 How many permutations can you get from the sequence

$$
\{1,2,3,4,5,6,7\}
$$

Question 20.3 How many variations with repetition can you get from the sequence

$$
\{1,2,3,4,5,6,7\}
$$

Question 20.4 How many variations with0ut repetition can you get from the sequence

$$
\{1,2,3,4,5,6,7\}
$$

Question 20.5 How many combinations can you get from the sequence

$$
\{1,2,3,4,5,6,7\}
$$

Question 20.6 Calculate the value of the fraction

$$
\frac{4!* 5!+3!* 4!}{5!* 6!}
$$

Question 20.7 Simplify the fraction

$$
\frac{n!+5!}{(n-3)}!
$$

Question 20.8 List all permutations of the set $5 a b$
Question 20.9 List all permutations of the digits of the three-digit number 987
Question 20.10 List all distinct numbers formed from the digits
Question 20.11 List all 3-element variations without repetitions selected from the 5-element set $\{a, b, c, d\} e$.

Question 20.12 How many chess pairs can be formed in a class of 25 students so that each student plays only once with each other student?

Question 20.13 How many possible outcomes are there in a numbers game if we choose 5numbersfrom 45 numbers?

## Chapter 21

## Descriptive statistics

The first and important stage of statistical studies is data collection and thei interpretation. The most important statistical data are provided each year by the Central Statistical Office (GUS) with its seat in Warsaw. They concern information about the population in Poland, data about the models in industry and agriculture, in economic and financial matters. These data constitute important part of information for the planning and administration of the state. In addition, statistical data are collected in polls with questions of particular importance. For example, in polls and forecasts in elections to the Sejm and in important administrative decisions in which the voice of the public is important. The collected statistical data are presented in tables and illustrated in diagrams. Various forms of diagrams are used. The most common diagrams are in the form of bars or circles with an indication of colors or numbers or percentages. Thus, diagrams are a simple method of presenting statistical data.

### 21.1 Examples of statistical data and diagrams

We write statistical data in tables with a description of their meaning and numerical values.
Example 21.1 The school complex include a kindergarten, primary school, middle school and high school. The table below summarizes the number of students

| Type of school | Number of students | Part of total | Percent |
| :--- | :--- | :--- | :--- |
| Kindergarten (KG) | 125 | $1 / 8 \mathrm{z} \mathrm{1000}$ | $12.5 \%$ |
| Primary School (PSCH) | 250 | $1 / 4 \mathrm{z} \mathrm{1000}$ | $25 \%$ |
| Junior High School (JHS) | 375 | $3 / 8 \mathrm{z} \mathrm{1000}$ | $37.5 \%$ |
| High School (HSCH | 250 | $1 / 4 \mathrm{z} \mathrm{1000}$ | $25 \%$ |
| Total | 1000 | $\frac{1}{8}+\frac{1}{4}+\frac{3}{8}+\frac{1}{4}=1$ | $100 \%$ |

In the diagrams below, in the form of bars and circles, there are charts of girls, boys and students in Kindergarten (KG), Primary School (PSCH), Middle School (JHS), and Liceuam
( HSCH ).
Legend: Girls are in the first column, boys are in the second column and the number of students is now in the third column. The charts are replicated for each of the four schools.


### 21.2 Mean value and median

Important parameters of statistical data are the mean value and the median value.
Arithmetic average. The arithmetic mean value of $n$ of numbers $a_{1}, a_{2}, \ldots, a_{n}$ is the number

$$
\text { Mean arithmetic }=\frac{a_{1}+a_{2}+\cdots+a_{n}}{n}
$$

Weighted arithmetic mean. A more general concept of the mean is the concept of a weighted arithmetic mean. Namely, let the weights be positive numbers $\rho_{1}, \rho_{2}, \cdots, \rho_{n}$ such that the sum

$$
\rho_{1}+\rho_{2}+\cdots+\rho_{n}=1, \quad \rho_{i}>0, \quad i=1,2, \ldots, n .
$$

Then we call the weighted average the following sum of products

$$
\text { Mean arithmetic weighted }=\rho_{1} a_{1}+\rho_{2} a_{2}+\cdots+\rho_{n} a_{n}
$$

Indeed, in a special case when the weights are equal

$$
\rho_{1}=\rho_{2}=\cdots=\rho_{n}=\frac{1}{n}
$$

Then the arithmetic mean is simply the arithmetic mean.
Median. For statistical data, we find their median, that is, the value that lies in the middle of the data. Namely, we first sort the data by ordering it from the smallest to the largest or from the largest to the smallest. Then the number equidistant from the beginning and the end of the ordered data is called mmedian. It may happen that there is no such single number, but there are two numbers next to each other that lie at the same distance, the first from the beginning and the second from the end. Then the median is their arithmetic mean.
Below, we explain it with examples.
Example 21.2 Let us consider the following data:
(i) $2,1,6,8,3,2,10,12,11$
(ii) $9,4,2,7,5,1,3,10,15,17,16$

Solution. The data of $2,1,5,8,3,2,10,12,11$ should be ordered in an ascending direction from the lowest to the larges

$$
1,2,2,3,6,8,10,11,12
$$

We notice that the number 6 is four positions away from the beginning and also four positions from the end. So the number 6 is the median of the data (i).
Solution (ii). The $0,-1,9,4,2,7,5,1,3,10,15,17.16$ data is ordered in the ascending direction from the lowest to the highest

$$
-1,0,1,2,3,4,5,7,9,15,16,17
$$

Note that the number 4 is five positions away from the start, and the number 5 is five positions away from the end. So we have two numbers in the middle of the data 4 and 5 . Then the median is their arithmetic mean, that is $\frac{4+5}{2}=4.5$. Answer. the median of the data (ii) is the number 4.5

### 21.2.1 Statistical data correlation

Consider two data sequences

$$
a=\left\{a_{1}, a_{2}, \ldots, a_{n}\right\}, \quad b=\left\{b_{1}, b_{2}, \ldots, b_{n}\right\}
$$

with the same number of $n$ elements.

Definicja 21.1 Correlation of statistical data

$$
a=\left\{a_{1}, a_{2}, \ldots, a_{n}\right\}, \quad b=\left\{b_{1}, b_{2}, \ldots, b_{n}\right\}
$$

we call the quotient

$$
\operatorname{Cor}(a, b)=\frac{a_{1} b_{1}+a_{2} b_{2}+\cdots+a_{n} b_{n}}{\sqrt{a_{1}^{2}+a_{2}^{2}+\cdots+a_{n}^{2}} \sqrt{b_{1}^{2}+b_{2}^{2}+\cdots+b_{n}^{2}}}
$$

We also write statistical data in their standardized form. Namely, let

$$
\begin{align*}
& \hat{a}=\left\{\hat{a}_{1}, \hat{a}_{2}, \ldots, \hat{a}_{n}\right\}=\frac{\left\{a_{1}, a_{2}, \ldots, a_{n}\right\}}{\sqrt{a_{1}^{2}+a_{2}^{2}+\cdots+a_{n}^{2}}} \\
& \hat{b}=\left\{\hat{b}_{1}, \hat{b}_{2}, \ldots, \hat{b}_{n}\right\}=\frac{\left\{b_{1}, b_{2}, \ldots, b_{n}\right\}}{\sqrt{b_{1}^{2}+b_{2}^{2}+\cdots+b_{n}^{2}}} \tag{21.1}
\end{align*}
$$

where

$$
\begin{aligned}
& \hat{a}_{1}=\frac{a_{1}}{\sqrt{a_{1}^{2}+a_{2}^{2}+\cdots+a_{n}^{2}}}, \quad \hat{b}_{1}=\frac{b_{1}}{\sqrt{b_{1}^{2}+b_{2}^{2}+\cdots+b_{n}^{2}}}, \\
& \hat{a}_{2}=\frac{a_{1}}{\sqrt{a_{2}^{2}+a_{2}^{2}+\cdots+a_{n}^{2}}}, \quad \hat{b}_{2}=\frac{b_{2}}{\sqrt{b_{1}^{2}+b_{2}^{2}+\cdots+b_{n}^{2}}}, \\
& \hat{a}_{n}=\frac{a_{n}}{\sqrt{a_{1}^{2}+a_{2}^{2}+\cdots+a_{n}^{2}}}, \quad \hat{b}_{n}=\frac{b_{n}}{\sqrt{b_{1}^{2}+b_{2}^{2}+\cdots+b_{n}^{2}}}
\end{aligned}
$$

Note that the statistical data (21.1) in normalized form meets the following conditions:

$$
\hat{a}_{1}^{2}+\hat{a}_{2}^{2}+\cdots+\hat{a}_{n}^{2}=1, ; \quad \hat{b}_{1}^{2}+\hat{b}_{2}^{2}+\cdots+\hat{b}_{n}^{2}=1
$$

Then the correlation between the data $a$ and $b$ and the correlation between the normalized data $\hat{a}$ and $\hat{b}$ is the same and determined as follows

Definicja 21.2 Correlation of statistical data

$$
\hat{a}=\left\{\hat{a}_{1}, \hat{a}_{2}, \ldots, \hat{a}_{n}\right\}, \quad \hat{b}=\left\{\hat{b}_{1}, \hat{b}_{2}, \ldots, \hat{b}_{n}\right\}
$$

we call the sum of the products

$$
\operatorname{Cor}(a, b)=\operatorname{Cor}(\hat{a} \hat{b})=\hat{a}_{1} \hat{b}_{1}+\hat{a}_{2} \hat{b}_{2}+\cdots+\hat{a}_{n} \hat{b}_{n}
$$

Example 21.3 Calculate the correlation between the data

$$
a=\{2,1,5,8\}, \quad b=\{4,3,9,3\}
$$

Solution. Substituting the data into the formula

$$
\begin{aligned}
& a_{1}=2, \quad a_{2}=1, \quad a_{3}=5, \quad a_{4}=8, \\
& b_{1}=4, \quad b_{2}=3, \quad b_{3}=9, \quad b_{4}=3
\end{aligned}
$$

we calculate the correlation coefficient

$$
\begin{aligned}
\operatorname{Cor}(a, b) & =\frac{a_{1} b_{1}+a_{2} b_{2}+\cdots+a_{n} b_{n}}{\sqrt{a_{1}^{2}+a_{2}^{2}+\cdots+a_{n}^{2}} \sqrt{b_{1}^{2}+b_{2}^{2}+\cdots+b_{n}^{2}}} \\
& =\frac{2 * 4+1 * 3+5 * 9+8 * 3}{\sqrt{2^{2}+1^{2}+5^{2}+8^{2}} \sqrt{4^{2}+3^{2}+9^{2}+3^{2}}}=0.769444,
\end{aligned}
$$

### 21.3 Variance and standard deviation

Variance $\sigma^{2}$ of the statistics

$$
a=\left\{a_{1}, a_{2}, \ldots, a_{n}\right\}
$$

is related to their arithmetic mean

$$
s=\frac{a_{1}+a_{2}+\cdots+a_{n}}{n}
$$

with the following formula:

$$
\sigma^{2}=\frac{\left(a_{1}-s\right)^{2}+\left(a_{2}-s\right)^{2}+\cdots+\left(a_{n}-s\right)^{2}}{n}
$$

We read sigma.
Standard Deviation $\sigma$ is the square root of the variance

$$
\sigma=\sqrt{\sigma^{2}}
$$

Example 21.4 Calculate the variances and standard deviation of the following data:

$$
\text { (i) } a=\{3,-1,8,4\}, \quad \text { (ii) } \quad b=\{12,4,8,6\} \text {. }
$$

Solution (i). The solution is a simple and direct substitution of data into formulas. First we calculate the mean value

$$
s=\frac{a_{1}+a_{2}+\cdots+a_{n}}{n}=\frac{3-1+8+4}{4}=3.5
$$

then we calculate the variance

$$
\begin{aligned}
\sigma^{2} & =\frac{\left(a_{1}-s\right)^{2}+\left(a_{2}-s\right)^{2}+\cdots+\left(a_{n}-s\right)^{2}}{n} \\
& =\frac{(3-3.5)^{2}+(-1-3.5)^{2}+(8-3.5)^{2}+(4-3.5)}{4}=10.31
\end{aligned}
$$

and standard deviation

$$
\sigma=\sqrt{\sigma^{2}}=\sqrt{10.31}=3.21131
$$

Solution (ii). Similarly, the solution of example (ii) is a direct substitution of data into formulas. First we calculate the mean value

$$
s=\frac{a_{1}+a_{2}+\cdots+a_{n}}{n}=\frac{12+4+8+6}{4}=\frac{30}{4}=7.5
$$

Solution (ii). Similarly, the solution of example (ii) is a direct substitution of data into formulas. First we calculate the mean value

$$
\begin{aligned}
\sigma^{2} & =\frac{\left(a_{1}-s\right)^{2}+\left(a_{2}-s\right)^{2}+\cdots+\left(a_{n}-s\right)^{2}}{n} \\
& =\frac{(12-7.5)^{2}+(4-7.5)^{2}+(8-7.5)^{2}+(6-7.5)^{2}}{4}=8.75
\end{aligned}
$$

and the standard deviation

$$
\sigma=\sqrt{\sigma^{2}}=\sqrt{10.31}=2.95804
$$

### 21.4 Questions

Question 21.1 Students from biology 20 students, chemistry 40 students, physics 50 students, computer science 50 students and mathematics 40 students signed up for the lecture in mathematics.

The table below presents information on the number of students

| Subject | Number of students | Part of the total | Percent |
| :--- | :--- | :--- | :--- |
| biology (BIO) | 20 | $1 / 10$ of 200 | $10 \%$ |
| chemistry (CHEM) | 40 | $1 / 5$ of 200 | $20 \%$ |
| physics(FIZ) | 50 | $1 / 4$ of 200 | $37.5 \%$ |
| IT (INF) | 50 | $1 / 4$ of 200 | $25 \%$ |
| math(MAT) | 40 | $1 / 5$ of 200 | $12.5 \%$ |
| Total | 200 | $\frac{1}{10}+\frac{1}{5}+\frac{1}{4}+\frac{1}{4}+\frac{1}{5}=1$ | $100 \%$ |

Use a ruler and a compass to draw diagrams in the form of bars and circles for students of biology (BIO), chemistry (CHEM), physics (FIZ), computer science (INF) and mathematics (MAT).
Also provide a description of the diagrams in the form of a legend.
Question 21.2 Calculate arithmetic average of the following data
(i) $2,11,16,18,13,12,10,12,11$
(ii) $29,24,22,27,25,21,23,10,15,17,16$

Question 21.3 Let us consider the following data:
(i) $2,11,16,18,13,12,10,12,11$
(ii) $29,24,22,27,25,21,23,10,15,17,16$

Find the median of the obove dat
Question 21.4 Find the correlation between data

$$
\{2,1,5,8,9,10\} \quad \text { and } \quad\{1,2,3,4,5,6\}
$$

Question 21.5 Calculate the variances and standard deviation of the following data:

$$
\text { (i) } \quad a=\{4,-3,8,9\}, \quad \text { (ii) } \quad b=\{2,5,8,7\} .
$$

Question 21.6 For the following data
(i) $4,6,8,10,12,14,16,18,20,22$
(ii) $19,17,15,13,11,9,7,5,3,1$
calculate the arithmetic mean the weighted arithmetic mean $\rho=\frac{1}{10}$ and the median
Question 21.7 Calculate the correlation between the data

$$
a=\{21,15,8,8\}, \quad b=\{1,3,6,9\}
$$

Question 21.8 Calculate the variances and standard deviation for the following data:

$$
\text { (i) } \quad a=\{5,1,6,4\}, \quad \text { (ii) } \quad b=\{10,6,2,1\} .
$$

## Chapter 22

## Introduction to the calculus of probability

### 22.1 Introduction

The foundations of the calculus of probability wewre created by Pascal (1623-1662 CE) and Fermat (1601-1665 AD) in the mid-17th century. In the eighteenth and nineteenth centuries, as important discovery was the law of large numbers by J. Bernoulli. Also works of A. Moivre, P. Laplas and S. Poisson. Chebyshev, Brown and Kolmogorov.

The theory of probability concerns the laws governing random phenomena, that is, phenomena whose course or result cannot be unequivocally predicted. This is because the course of a random phenomenon is usually influenced by many reasons, only some of which can be controlled.
The results of random phenomena (experiences) are called random events.
If we repeat $n$ times and in these $n$ experiments exactly $k$ times we observe the result of $A$, then the number

$$
\frac{k}{n}
$$

is called frequency of the $A$ random event in a series of $n$ experiences.
In mass phenomena, the frequencies of the occurrence of each random event have the property that with the increase of the number $n$, these frequencies "stabilize" more and more "close to" a certain number characteristic for the experiment.

Generally, in an experiment repeated under the same conditions for each event the "characteristic" number in case of two results experiment, as in the amonet tossing, is equal to $\frac{1}{2}$. In this case the frequencies

$$
\frac{k_{n}}{n} \rightarrow \frac{1}{2}, \quad n=1,2,3, \cdots
$$

approach to $\frac{1}{2}$, when the number of experiences $n \rightarrow \infty$ goes to infinity.

Example 22.1 For example, tossing a symmetrical coin of $n=100$, 200 or more times, we will see about half tails and about half heads.

The following results in 100 and 200 coin tosses indicate a stabilization of frequency the "near" $\frac{1}{2}$ for the $n \geq 200$ experience.

> Table

| Number of toss | Number of tails | Number of heads | Frequency | Frequency- |
| :---: | :---: | :---: | :---: | :---: |
| n | k | $\mathrm{n}-\mathrm{k}$ | $\frac{k}{n}$ | $\frac{n-k}{n}-$ |
| 100 | 61 | 39 | 0.61 | $0.39-$ |
| 200 | 102 | 98 | $0.51-$ | 0.41 |

### 22.2 Elementary random events

In each random experiment we can distinguish the simplest results called elementary events. The set of all elementary events is called probability space marked with the letter $\Omega$

Example 22.2 By tossing a coin, two results are possible tails or the heads and there are not other options. In this example the probality space

$$
\Omega=\left\{\omega_{1}, \omega_{2}\right\}
$$

consits of two elementary events $\omega_{1}, \omega_{2}$,
Heuristic definition of probability. The probability is the number around which the frequency stabilizes when the number of random experiences increases.

Below on examples we will explain the frequency stabilization around the probability number.

Coin toss. We start with the simplest coin toss experience. When tossing a coin, there are two possible tails results, or there are no other possibilities for an head.

We ask what chances we have for tails to appear?
Of the two possible results one is for tails and other for heads. Thus, the chance of a tails to appear is $\frac{1}{2}$ and an head to appear is also equal to $\frac{1}{2}$.

If the result of the appearance of tails is denoted by the letter $A$, and the result of the appearance of heads with the letter $B$, then the probability of the appearance of tails is denoted by the symbol $P(A)$, and the probability of the appearance of an heads with the symbol $P(B)$. We write

$$
P(A)=\frac{1}{2}, \quad P(B)=\frac{1}{2}
$$

The probability of $\frac{1}{2}$ means that we expect half tails and half heads in a large number of casts.

Example 22.3 Count how many times tail appears, and how many times head appears, in 10 coin tosses

Suppose tails appeared on the first, fifth, eighth and tenth flushes, 4 times in total, and the head appeared 6 times.
We denote the event of the appearance of tails with the letter A, the event of the appearance of the heads with the letter B.
Calculate the frequency of tails appearing

$$
\text { Frequency }(A)=\frac{4 \text { events favorable }}{10 \text { possibleevents }}=\frac{4}{10}=\frac{2}{5}
$$

Similarly, we calculate the frequency of the appearance of head as the ratio of the 6 favorable events to all 10 possible events

$$
\operatorname{Frequency}(B)=\frac{6 \text { favorableevents }}{10 \text { events possible }}=\frac{6}{10}=\frac{3}{5}
$$

Example 22.4 There were 20 students in the class. Each student tossed a symmetrical coin 50 times. Below in the table the frequencies of tails falling in 100 and 1000 throws are given.

| Number of toss | Number of tails | Number of heads | Frequency | Frequency- |
| :---: | :---: | :---: | :---: | :---: |
| n | k | $\mathrm{n}-\mathrm{k}$ | $\frac{k}{n}$ | $\frac{n-k}{n}-$ |
| 100 | 54 | 46 | 0.54 | $0.45-$ |
| 1000 | 517 | 483 | $0.517-$ | 0.483 |

We can see that the frequency of tails appearing per 100 coin tosses is 0.54 , while tails appearing on 1000 tosses is equal to 0.517 . The number of 0.517 per 1000 flips is closer to the characteristic number of 0.5 than the number of 0.54 per 100 coin flips, in this example. This observed regularity that the random event frequency is "stable" around some constant value when the number of repetitions of the random experiment is large, underlies the concept of probability.

Below we explain the following concepts concerning random events

- disjoint random events
- certain random events
- impossible random events
- The probability of events

Let's denote the event of the appearance of tails with the letter A, the event of the appearance of the head appearing in the coin toss with the letter B . These events $A$ and $B$ are disjoint.
The $A$ and $B$ events are mutually exclusive, because $A$ event excludes $B$ event.
We write alternative of $A$ or $B$ in formula

Let us calculate the frequency of the alternative of disjoint random events

$$
\operatorname{Frequency}(A \cup B)=\operatorname{Frequency}(A)+\operatorname{Frequency}(B)=\frac{2}{5}+\frac{3}{5}=1
$$

Thus frequency of the alternative of $A$ and $B$ events is equal to the sum of the frequencies.
Similarly, we calculate the probability of the alternative of disjoint random events $A$ and $B$.

$$
P(A \cup B)=P(A)+P(B)=\frac{1}{2}+\frac{1}{2}=1
$$

It means the probability of alternative of the disjoint random events $A$ and $B$ equals sum of the probabilities of $A$ random event and $B$ random event.

In a coin toss, the appearance of a tails or a head is a certain random event, the probability of which is equal to 1 .

On the other hand, the event that no tails or heads appear in the coin toss is impossible.
Therfore probability of the impossible random event is equal to 0
Example 22.5 Let us consider the experience of throwing a dice in the shape of a regular cube, on the sides of which there are meshes from 1 to 6 .


Visiable meshes on sides of the dice in the shape of cube
In a dice throw there are possible the following readings:
1 meshe, 2 meshes, 3 meshes, 4 meshes, 5 meshes and 6 meshes
which corespond to the following elmetary random events
$\omega_{1}$ event, when there is 1 meshes
$\omega_{2}$ event, when 2 meshes appear
$\omega_{3}$ event, when 3 meshes appear
$\omega_{4}$ event, when 4 meshes appear
$\omega_{5}$ event when 5 meshes appear
$\omega_{6}$ event, when 6 meshes appear
Thus, in one dice roll there are 6 possible outcomes
1 meshes, 2 meshes, 3 meshes, 4 meshes, 5 meshes, 6 meshes
Each side of a dice have the same chance of appearing one of 6 possible outcomes. Their probability are equal to $\frac{1}{6}$ as we write.

$$
\begin{array}{ll}
P\left(\omega_{1}\right)=\frac{1}{6}, & P\left(\omega_{2}\right)=\frac{1}{6} \\
P\left(\omega_{3}\right)=\frac{1}{6}, & P\left(\omega_{4}\right)=\frac{1}{6} \\
P\left(\omega_{5}\right)=\frac{1}{6}, & P\left(\omega_{6}\right)=\frac{1}{6}
\end{array}
$$

Suppose we throw $N=100$ times dice and we recod the results below.
The event $\omega_{1}$ appeared 17 times, i.e. 1 meshes appeared 17 times
The event $\omega_{2}$ appeared 16 times, i.e. 2 meshes appeared 16 times
The event $\omega_{3}$ appeared 17 times, i.e. 3 meshes appeared 17 times The event $\omega_{4}$ appeared 18 times, i.e. 4 meshes appeared 18 times Event $\omega_{5}$ popped 15 times, it would get 5 meshes popped 15 times The event $\Omega_{6}$ appeared 17 times, i.e. 6 meshes appeared 17 times

In this experience, the frequency of one out of six results are below:

$$
\begin{aligned}
& \text { Frequency }\left(\omega_{1}\right)=\frac{17 \text { favorable events }}{100 \text { events possible }}=\frac{17}{100}=0.17 \\
& \text { Frequency }\left(\omega_{2}\right)=\frac{16 \text { favorable events }}{100 \text { events possible }}=\frac{16}{100}=0.16 \\
& \text { Frequency }\left(\omega_{3}\right)=\frac{17 \text { favorable events }}{100 \text { events possible }}=\frac{17}{100}=0.17 \\
& \text { Frequency }\left(\omega_{4}\right)=\frac{18 \text { favorable events }}{100 \text { events possible }}=\frac{18}{100}=0.18 \\
& \text { Frequency }\left(\omega_{5}\right)=\frac{15 \text { favorable events }}{100 \text { events possible }}=\frac{15}{100}=0.15 \\
& \text { Frequency }\left(\omega_{6}\right)=\frac{17 \text { favor events }}{100 \text { events possible }}=\frac{17}{100}=0.16
\end{aligned}
$$

The probability space

$$
\Omega=\left\{\omega_{1}, \omega_{2}, \omega_{3}, \omega_{4}, \omega_{5}, \omega_{6}\right\}
$$

is the set of all disjoint - exclusive elementary random events.
Therefore, the alternative to all elementary events

$$
A=\omega_{1} \cup \omega_{2} \cup \omega_{3} \cup \omega_{4} \cup \omega_{5} \cup \omega_{6}
$$

is a certain event.
The probability of a certain $A$ event is 1 , we write

$$
P(A)=P\left(\omega_{1} \cup \omega_{2} \cup \omega_{3} \cup \omega_{4} \cup \omega_{5} \cup \omega_{6}\right)=1
$$

Also, the frequency of a certain event is equal to 1 because it is equal to the sum of the disjoint elementary frequencies

$$
\text { Frequency }(\omega)=\frac{17}{100}+\frac{16}{100}+\frac{17}{100}+\frac{18}{100}+\frac{15}{100}+\frac{17}{100}=1
$$

### 22.3 Equally probable events

Let us observe that probability of appearence of tails or heads in the toss of a coin is equal to $\frac{1}{2}$. Also, in the dice roll results of

1 meshes, 2 meshes, 3 meshes, 4 meshes, 5 meshes, 6 meshes
are equally probable and their probability of appearence is equal to $\frac{1}{6}$.
Below we present Laplace definitions of probability for equal-probable random events.
Laplace definition of probability. If for a given experience the set of elementary random events consists of N equally probable events then the probability of $A$ random event is equal to

$$
P(A)=\frac{n}{N}
$$

where $n$ is the number of events favoring $A .{ }^{1}$ Let us consider the following egzample
Example 22.6 One card was drawn at random from the 52-card deck. Calculate the probability of $P(A)$ of the following random events:
(i) The A event is to pull an ace.
(ii) The $A$ event is to pull a spades.
(iii) The $A$ event is to pull a spades or clubs.

Solution (i). The set of elementary events consists of $N=52$ equal events.
The probability of drawing each card is the same and equal to $\frac{1}{52}$.
Since there are 4 aces in the deck, the number of events that favor an ace is $n=4$. So the probability of drawing an ace is

$$
P(A)=\frac{4}{52}=\frac{1}{13}
$$

Solution (ii). The set of elementary events consists of $N=52$ equal events,
The probability of drawing each card is the same and equal to $\frac{1}{52}$.
Since there are 13 spades in the deck, the number of events favoring a spades is $n=13$. Thus, the probability of removing a spades is

$$
P(A)=\frac{13}{52}=\frac{1}{4}
$$

Solution (iii). The set of elementary events consists of $N=52$ of equal probability. The

[^36]probability of drawing each card is the same and equal to $\frac{1}{52}$.
Since there are 13 spades and 13 clubs in the deck, the number of events that favor drawing a spades or a clubs is
$$
n=13+13=26
$$

So the probability of drawing a spades or a clubs is

$$
P(A)=\frac{26}{52}=\frac{1}{2}
$$

Let us consider one more example of random events.
Example 22.7 There are 250 girls and 200 boys on the school list. One name was randomly selected from the list. What is the probability of $A$ that this is (a) a girl, (b) a boy

Solution (a). Total on the list is $250+200=450$ students. The probability space

$$
\Omega=\left\{\omega_{1}, \omega_{2}, \omega_{3}, \cdots, \omega_{450}\right\}
$$

consists of $N=450$ elementary random events. Each of these elementary events $\omega_{i}, i=$ $1,2,3, \cdots, 450$ consists of pulling one first and last name from a list of 450 students.
The number of events conducive to the fact that it is a girl is equal to $n=250$. Probability of being a girl

$$
P(A)=\frac{250}{450}=\frac{5}{9}
$$

Solution (b). We similarly calculate the probability of selecting a boy's name from the list. Total on the list is $250+200=450$ students. Set of all elementary events

$$
\Omega=\left\{\omega_{1}, \omega_{2}, \omega_{3}, \cdots, \omega_{450}\right\}
$$

consists of $N=450$, elementary events. Each of these elementary events $\Omega_{1}, i=1,2,3, \cdots$, 450 consists in drawing one name from a list of 450 students.
The number of events favoring this being a boy is $n=200$. The probability that it is a boy

$$
P(A)=\frac{200}{450}=\frac{4}{9}
$$

### 22.4 Elementary and composite random events

Results of operations such as alternative, conjction, and difference performed on on random events we call composite random events
Thus, by performing these operations on the set of elementary events, we obtain composite random events.
For example, in a dice roll experiment all elementary events are listed in the probability space below

$$
\Omega=\left\{\omega_{1}, \omega_{2}, \omega_{3}, \omega_{4}, \omega_{5}, \omega_{6}\right\}
$$

In the probability space $\Omega$ we distinguish subsets as composied random events:

- Certain random event specified as the set

$$
\Omega=\left\{\omega_{1}, \omega_{2}, \omega_{3}, \omega_{4}, \omega_{5}, \omega_{6}\right\}
$$

Of course each of elementary random events $\omega_{i}, 1,2,3,4,5,6$ in dice rolling occurs with probability

$$
P\left(\omega_{i}\right)=\frac{1}{6} . \quad i=, 1,2,3,4,5,6
$$

Therefore the probability of the alternative is equal 1 , we write

$$
P\left(\omega_{1} \cup \omega_{2} \cup \omega_{3} \cup \omega_{4} \cup \omega_{5} \cup \omega_{6}\right)=1
$$

- The expected result of random event $A$ is number of meshes. This random event occurs if $\omega_{2}$ or $\omega_{4}$ or $\omega_{6}$ occurs. That is, when the alternative

$$
\omega_{2} \cup \omega_{4} \cup \omega_{6}
$$

is true Thus, $A$ is determined by the subset

$$
A=\left\{\omega_{2}, \omega_{4}, \omega_{6}\right\} \subset \Omega
$$

Elementary events $\omega_{2}, \omega_{4}, \omega_{6}$ favor the occurrence of the $A$ event.
The probability of this event

$$
P(A)=\frac{3}{6}=\frac{1}{2}
$$

- The expected result of $A$ is a number of meshes less than 3. This random event occurs if the dice roll is one or two if the alternative is true

$$
\omega_{1} \cup \omega_{2}
$$

So the $A$ event is determined by the subset

$$
A=\left\{\omega_{1}, \omega_{2}\right\} \subset \Omega
$$

Elementary events $\omega_{1}, \omega_{2}$ favor $A$
The probability of this event happening

$$
P(A)=\frac{2}{6}=\frac{1}{3}
$$

- The expected result of random event $A$ is the number of meshes greater than 3 . This random event occurs if it occurs $\omega_{4}$ or $\omega_{5}$ or $\omega_{6}$ when $\omega_{4} \cup \omega_{5} \cup \omega_{6}$ Thus random event $A$ is specified by the subset

$$
A=\left\{\omega_{4}, \omega_{5}, \omega_{6}\right\} \subset \Omega
$$

Each of the $\omega_{4}, \omega_{5}, \omega_{6}$ events favors the random event $A$
The probability of this event

$$
P(A)=\frac{3}{6}=\frac{1}{2}
$$

### 22.5 Operations on random events

The random event opposite to the given random event $A$ is the set of those elementary events of the probability space which do not match the random event $A$. The random event opposite to the random event $A$ is denoted here by the symbol $A^{\prime}$. The relation that the element $x$ belongs to the set $\Omega$, we write $x \in \Omega$. Also the relation that $x$ does not belong to the set $\Omega$, we wrte $x \notin \Omega$.
We understand random events as subsets of the set of elementary events $\Omega$. Operations such as sum, product and difference on sets are also applied to on random events, because they are operations to subsets of the set of all elentary events.

### 22.6 Opposite random event

The opposite event to the $A$ event is marked with the symbol $A^{\prime}$. The opposite of $A^{\prime}$ occurs when no random event $A$ has occurred.
For example, let $A$ result in an even number of dice being rolled.
Then

$$
A=\left\{\omega_{2}, \omega_{4}, \omega_{6}\right\}
$$

is a subset of the set of elementary events

$$
\Omega=\left\{\omega_{1}, \omega_{3}, \omega_{3}, \omega_{4}, \omega_{5}, \omega_{6}\right\}, \quad A \subset \Omega
$$

The opposite of $A^{\prime}$ will occur if there is an odd number of meshes

$$
A^{\prime}=\left\{\omega_{1}, \omega_{3}, \omega_{5}\right\}
$$

The $A^{\prime}$ subset is also a subset of the set of elementary events

$$
\Omega=\left\{\omega_{1}, \omega_{3}, \omega_{3}, \omega_{4}, \omega_{5}, \omega_{6}\right\}, \quad A^{\prime} \subset \Omega
$$

Note that the opposite event is equal to the difference of the set of all elementary events

$$
\Omega=\left\{\omega_{1}, \omega_{3}, \omega_{3}, \omega_{4}, \omega_{5}, \omega_{6}\right\}
$$

and rando event wydarzenia $A$
Hence we have

$$
A^{\prime}=\Omega-A=\left\{\omega_{1}, \omega_{3}, \omega_{3}, \omega_{4}, \omega_{5}, \omega_{6}\right\}-\left\{\omega_{2}, \omega_{4}, \omega_{6}\right\}=\left\{\omega_{1}, \omega_{3}, \omega_{5}\right\}
$$

### 22.7 Alternative of random events

Alternative to $A$ and $B$ random events is event

$$
C=A \cup B
$$

which happens if and only if either $A$ or $B$ occurs.
For example, in a dice roll let let $A$ be a result greater than 5 and $B$ be a result less than 2.
Sure, the event

$$
A=\omega_{6}
$$

while the event

$$
B=\omega_{1}
$$

An alternative to these events is a subset

$$
C=A \cup B=\left\{\omega_{1}, \omega_{6}\right\}
$$

a set of elementary events $\Omega$. we write

$$
C=A \cup B \subset \Omega
$$

### 22.8 Conjaction of random events

The event of $A$ and $B$ is the event

$$
D=A \cap B
$$

which occurs if and only if the $A$ event occurs and the $B$ event occurs simultaneously.
For example, let $A$ be a number of meshes greater than 3 , let $B$ be a number of meshes less than 5.
Sure, the $A$ event specifies a subset

$$
A=\left\{\omega_{4}, \omega_{5}, \omega_{6}\right\} \subset \Omega
$$

and the $B$ event specifies a subset

$$
B=\left\{\omega_{1}, \omega_{2}, \omega_{3}, \omega_{4}\right\} \subset \Omega
$$

The conjaction $A \cap B$ is an event

$$
D=A \cap B=\left\{\omega_{4}, \omega_{5}, \omega_{6}\right\} \cap\left\{\omega_{1}, \omega_{2}, \omega_{3}, \omega_{4}\right\}=\left\{\omega_{4}\right\}
$$

which consists of the same elementary events simultaneously belonging to $A$ and $B$.

### 22.9 Disjoint random events

The $A$ and $B$ events turn off if their conjunction is the empty set. That is, $A \cap B=\emptyset$.
For example, in a die roll experiment, let an event

$$
A=\left\{\omega_{1}, \omega_{6}\right\}
$$

means appearance of one or six stitches, while event

$$
B=\left\{\omega_{3}, \omega_{4}, \omega_{5}\right\}
$$

means there are three or four or five stitches to appear. These events are disjointed as they have a conjunction

$$
A \cap B=\left\{\omega_{1}, \omega_{6}\right\} \cap\left\{\omega_{3}, \omega_{4}, \omega_{5}\right\}=\emptyset
$$

is the empty set of events.

### 22.10 Difference of random event

The difference between random events $A$ and $B$ is the random event

$$
E=A-B
$$

which happens when $A$ occurs and $B$ does not occur.
For example, let in dice rolling random event $A$ be an even number of meshes and random event $B$ be a number of meshes divisible by 3 .
Sure, the $A$ event happens when we pull an element of the subset

$$
\hat{\omega}=\left\{\omega_{2}, \omega_{4}, \omega_{6}\right\} \subset \Omega
$$

However, the $B$ event does not occur unless an element of the subset is selected of the set $\Omega$

$$
\left\{\omega_{3}, \omega_{6}\right\} \subset \Omega
$$

Thus the difference of $A$ and $B$ random events is the random event

$$
E=A-B
$$

which occurs when we select an element of the $A$ subset and at the same time we do not select an element of the $B$ subset. Thus the difference of random events $A-B$ equals to random event $E$ given by formula

$$
E=A-B=\left\{\omega_{2}, \omega_{4}, \omega_{6}\right\}-\left\{\omega_{3}, \omega_{6}\right\}=\left\{\omega_{2}, \omega_{4}\right\} \subset \Omega
$$

### 22.11 Examples of random events

Below we present examples of operations on random events.
Example 22.8 Consider the set $\Omega$ of twnety natural numbers

$$
\Omega=\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17.18,19,20\}
$$

Calculate the probability of drawing a number from this set that is divisible by 5.
Solution Collection $\Omega$ is the probability space of $N$ possible random events.

Favorable random events of drawing a number divisible by 5

$$
A=\{5,10,15,20\}
$$

consists of $k=4$ numbers.
Thus, the probability of drawing a number divisible by 5 from the set $\Omega$ of all possible events is equal

$$
P(A)=\frac{k}{N}=\frac{4}{20}=\frac{1}{5}
$$

Answer. The probability of drawing from the set $\Omega$ a number divisible by 5 is equal to $\frac{1}{5}$.
Example 22.9 We choose two numbers at random from the set

$$
\{1,2,3,5,7,9\}
$$

Calculate the probability of getting two numbers where at least one is divisible by 3.
Solution (22.9). In this example, the space of elementary events consists of all pairs of numbers given in the array

$$
\Omega=\left\{\begin{array}{lllll}
(1,1), & (1,2), & (1,3), & (1,5), & (1,7), \\
(2,1), 9), \\
(3,1), & (2,2), & (2,3), & (2,5), & (2,7), \\
(2,9), \\
(5,1), & (5,2), & (5,3), & (3,5), & (3,7), \\
(3,5), & (5,7), & (5,9), \\
(7,1), & (7,2), & (7,3), & (7,5), & (7,7), \\
(7,9), \\
(9,1), & (9,2), & (9,3), & (9,5), & (9,7), \\
(9,9),
\end{array}\right\}_{N=6 \times 6}
$$

Thus, the set of all elementary random events consists of $N=36$ possible events. The favoring random events are those pairs of numbers selected from the array (22.9) in which at least one number is divisible by 3 are listed in the array

$$
A=\left\{\begin{array}{rrrr} 
& (1,3), & (1,9) \\
& (2,3), & & (2,9), \\
(3,1), & (3,2), & (3,3), & (3,5), \\
& (5,3), & (3,7), & (3,9), \\
& (7,3), & & (5,9), \\
(9,1), & (9,2), & (9,3), & (9,5), \\
& (9,7), & (9,9),
\end{array}\right\}_{k=20}
$$

which consists of $k=20$ pairs where at least one of the numbers is divisible by 3 .
Thus, the probability of drawing two numbers from the set of $\{1,2,3,5,7,9\}$ of which one is divisible by 3 is equal to

$$
P(A)=\frac{k}{N}=\frac{20}{36}=\frac{5}{9}
$$

Example 22.10 From the set of numbers

$$
\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17.18,19,20\}
$$

we choose randomly one number. Find the probability of getting a number that is divisible by 2 and by 3.

Solution (22.10). Set of all possible random events

$$
\Omega=\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17.18,19,20\}
$$

consists of $N=20$ elementary random events.
The set of random events favoring the

$$
A=\{6,12,18\}
$$

that a drawn number is divisible by 2 and by 3 , consists of $k=3$ numbers.
Therefore probability of a number divisible by 2 and by 3 in the set $\Omega$ is equal to

$$
P(A)=\frac{k}{N}=\frac{3}{20}
$$

Example 22.11 From the set of numbers

$$
\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17.18,19,20\}
$$

we choose randomly one number. Calculate the probability to obtain a number smaller than 6.

Solution (22.11). Set of all possible random events

$$
\Omega=\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17.18,19.20\}
$$

consists of $N=20$ elementary random events.
The set of random events favoring the

$$
A=\{1,2,3,4,5\}
$$

that the drawn number is less than 6 , consists of $k=5$ numbers.
Therefore the probability of drawing a number less than 6 from the set $\Omega$ is equal

$$
P(A)=\frac{k}{N}=\frac{5}{20}=\frac{1}{4}
$$

Example 22.12 .
(i) Give the set of all elementary random events in the experiment by throwing two dice. (ii) Calculate the probability of random event $A$ when tossing two dice to get sum of meshes that is even
(iii) Calculate the probability of random event A when for one toss two dice to get sum of meshes that is odd.

Solution (i) By throwing first and secound dice we get 36 pairs of meshes as all possible results listed in the array below

$$
\Omega=\left\{\begin{array}{llllll}
(1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\
(2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6), \\
(3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6), \\
(4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6), \\
(5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6), \\
(6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6),
\end{array}\right\}_{N=6 \times 6}
$$

Solution (ii). In a two dice roll, the total of meshes is even number in two cases

1. If both the first and the second results are even numbers
2. On both the first and the second dice are uneven numbers of meshes.

In the space of all possible random events there are $N=6 * 6=36$ pairs.
Let us observe that there are

$$
k=3 * 6=18
$$

of favorable random events which are listed in the array below

$$
\omega=\left\{\begin{array}{lll}
(1,1), & (1,3), & (1,5), \\
(2,2), & (2,4), & (2,6), \\
(3,1), & (3,3), & (3,5), \\
(4,2), & (4,4), & (4,6), \\
(5,1), & (5,4), & (5,5), \\
(6,2), & (6,4), & (6,6),
\end{array}\right\}
$$

Now we can calculate the probability of to occure random event $\mathrm{A}^{\prime}$

$$
P(A)=\frac{k}{N}=\frac{18}{36}=\frac{1}{2}
$$

Solution (iii). In a two dice roll, the total of meshes is odd number in two cases

1. If on the first dice there is an even number of meshes but on secound dice thre is odd number of meshes
2. If on the first dice there is an odd number of meshes but on secound dice there is an even number of meshes
In the space of all possible random events there are $N=6 * 6=36$ pairs. (see solution (i)) Let us observe that there are

$$
k=3 * 6=18
$$

of favorable random events to event $A$ ' which are listed in the array below

$$
\omega^{\prime}=\left\{\begin{array}{lll}
(1,2), & (1,4), & (1,6), \\
(2,1), & (2,3), & (2,5), \\
(3,2), & (3,4), & (3,6), \\
(4,1), & (4,3), & (4,5), \\
(5,2), & (5,4), & (5,6), \\
(6,1), & (6,3), & (6,5),
\end{array}\right\}
$$

The number of all favorable random events to $\mathrm{A}^{\prime}$ is equl $k=3 * 6=18$, that is equal to number of elements of the array $\omega$.

Thus, the probability of the random event A' that the sum of meshes on two dice as result of one toss of two dice is odd number is equal

$$
P\left(A^{\prime}\right)=\frac{k}{N}=\frac{18}{36}=\frac{1}{2}
$$

Example 22.13 (i) Calculate the probability of drawing in a number lottery from the set of numbers $\{1,2, \ldots, 49\}$ (i) six numbers, (ii) five numbers, no repetition.

Solution (i). Let us consider random event A as a set consisting of six natural numbers

$$
A=\left\{n_{1}, n_{2}, n_{3}, n_{4}, n_{5}, n_{6}\right\}
$$

where numbers $n_{i} \in A, i=1,2,3,4,5,6$ are elements of set A .
In this exercise, the elementary event space is the set of all six-element different subsets of the set of forty-nine number lotteries. Note that these subsets of numbers are six-element combinations selected from forty-nine lottery numbers. The number N of all six-element combinations is calculated from the formula ${ }^{2}$

$$
N=\binom{49}{6}=\frac{49!}{6!*(49-6)!}=\frac{49!}{6!* 43!}=13983816
$$

Because only one of 13983816 random events win, therefore probability of win is equal to

$$
P(A)=\frac{1}{13983816}=0,00000007151=0,0000071 \%
$$

[^37]Solution (ii). Let the six numbers $\left\{k_{1}, k_{2}, k_{3}, k_{4}, k_{5}, k_{6}\right\}$ be the result of a lottery draw. We know from solution (i) that the set of elementary events

$$
\Omega=\left\{\omega=\left\{n_{1}, n_{2}, n_{3}, n_{4}, n_{5}, n_{6}\right\}, 1 \leq n_{i} \leq 49, i=1,2,3,4,5,6 .\right\}
$$

contains $N=\binom{49}{6}=13,983,816$ items.
Let us denote by $A$ the event that the player selected the numbers $\left\{s_{1}, s_{2}, s_{3}, s_{4}, s_{5}, s_{6}\right\}$ among which five numbers are hit. That is, in this set of numbers

$$
\left\{s_{1}, s_{2}, s_{3}, s_{4}, s_{5}, s_{6}\right\}
$$

five numbers are from the set $\left\{k_{1}, k_{2}, k_{3}, k_{4}, k_{5}, k_{6}\right\}$.
Now let's count how many events favoring $A$. First, note that one number missed could be $k_{1}$ or $k_{2}$ or $k_{3}$ or $k_{4}$ or $k_{5}$ or $k_{6}$. Thus, one of the five hit numbers can be replaced with a missed number to get another five hit numbers. This conversion can be done for $\binom{6}{1}=6 \operatorname{six}$ ways.
Thus we find six different sixes as events favoring the $A$ event.
Also, the remaining 43 numbers were not drawn from the lottery ticket. Each of these 43 not picked in the game can be exchanged for one of the six missed by the player. Thus, the event favoring $A$ draws the correct five number is $6 * 43=\mathrm{k}=258$. This means that the probability of drawing the correct 5 numbers out of the drawn six numbers is equal to

$$
P(A)=\frac{k}{N}=\frac{258}{13983816}=0,00001845
$$

### 22.12 Questions

Question 22.1 If you toss a coin 100 times, heads appear 45 times and tails appear 55 times. Calculate the frequency of heads and tails.

Question 22.2 The random experiment involves tossing two coins. Calculate the probability of two heads appearing in 50 tosses.

Question 22.3 A random experiment involves rolling the dice. Calculate the probability of getting a sum of 4 in 100 throws.

Question 22.4 Winning in a numbers game involves drawing four numbers out of forty numbers from 1 to 40. Calculate the probability of winning.

Question 22.5 .
Make 50 coin tosses and count the number of heads in 10, 20, 30, 40, 50 throws.
Calculate the frequency of occuring tails and heads for 10, 20, 30, 40, 50 in throws.
Find probability of appearing tails and heads.
Question 22.6 Make 10 throws with two of the same coin.
Calculate the frequency of the appearance of heads and tails in 10 throws.
Question 22.7 Consider the probabilistic model of tossing two coins once.
(i) Calculate the probability of two heads.
(ii) Calculate the probability of the appearance of head and tail.

Question 22.8 Consider the probabilistic model of throwing two dice only once.
(i) Calculate the probability of two identical numbers of meshes.
(ii) Calculate the probability of the occurrence of the number of meshes the sum of which
is 4.
(iii) Calculate the probability of the occurrence of the number of meshes the sum of which is 12.

Question 22.9 From the set of numbers

$$
\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17.18,19,20\}
$$

choose randomly one number. Calculate the probability to get a number that is divisible by 4.

Question 22.10 From the set of numbers

$$
\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17.18,19,20\}
$$

choose randomly one number. Calculate the probability of getting a number that is divisible by 3 and by 4.

Question 22.11 From the set of numbers
$\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17.18,19,20\}$
choose randomly one number. Calculate the probability to obtain a number less than 9.
Question 22.12 Consider the elementary space of random events

$$
\Omega=\left\{\omega_{1}, \omega_{2}, \omega_{3}, \omega_{4}, \omega_{5}, \omega_{6}\right\}
$$

(i) Calculate the alternative and conjanction of random events

$$
A=\left\{\omega_{1}, \omega_{6}\right\}, \quad B=\left\{\omega_{2}, \omega_{6}\right\}
$$

(ii) Find the opposite random events of $A^{\prime}$ and $B^{\prime}$ to random events $A$ and $B$.
(iii) Calculate the probabilities of appearing of random events $A^{\prime}$ and $B^{\prime}$.


[^0]:    ${ }^{1}$ Algorithms take form of algebraic expressions

[^1]:    ${ }^{1} \mathrm{E}[\mathrm{x}]$ entire of $x$ means all of the number x

[^2]:    ${ }^{2} \mathrm{E}[\mathrm{x}]$ entire of $x$ means the whole of x , not grear then x
    ${ }^{3} \mathrm{E}[\mathrm{x}]$ integer part of $x$.

[^3]:    ${ }^{4}$ The simple Euclid algorithm is successfully used in computational systems. It is easy to write code in programming languages.

[^4]:    ${ }^{5} \mathrm{GCD}(\mathrm{x}, \mathrm{y})$ is the greatest coefficient of x and y .

[^5]:    ${ }^{1}$ Here binary numbers $(10)_{2}=10,110=(110)_{2}, 1010=(1010)_{2}$ etc...;

[^6]:    ${ }^{2}$ Here we omit the bracket $10 \equiv\left(10_{2}\right)$

[^7]:    ${ }^{3}$ Octal numbers we write $\left(\alpha_{0}, \alpha_{1}, \cdots, \alpha_{n-1}\right)_{8}$ in parentheses with the id at the bottom 8.

[^8]:    ${ }^{4}$ Here the octal numbers $(1)_{8}=1,(2)_{8}=2,(3)_{8}$ etc $\ldots ;$ we write with parentheses

[^9]:    ${ }^{5}$ Here we omit the bracket $2 \equiv(2)_{8}$ perform operations on octal numbers

[^10]:    ${ }^{1}$ We call identities an equality that is true for all parameter values

[^11]:    ${ }^{1}$ Next we use simplified notation $y$ instead of $y(x)$

[^12]:    ${ }^{2}$ Here we use the simplified notations $y=y(x), y_{0}=y\left(x_{0}\right), y_{1}=y\left(x_{1}\right)$

[^13]:    ${ }^{1}$ Read $D$ is the set of all real numbers from the interval $(-\infty, \infty)$

[^14]:    ${ }^{1}$ No concept of product or ratio of points
    ${ }^{2}$ The coordinates $v_{1}, v_{2}$ of the free vector $\vec{v}=\left[v_{1}, v_{2}\right]$ are written in square brackets.
    ${ }^{3} \mathrm{~A}$ free vector is defined by its length, direction and sense, does not depend on the position on plane.

[^15]:    ${ }^{4}$ The scalar value is a number

[^16]:    ${ }^{5}$ The segment starts at point $A$ and at the end in point $B$ denote by the symbol $[A, B]$

[^17]:    ${ }^{6}$ The lengths of segments $[A, B],[A, C],[B, C]$ are marked with the letters $a, b, c$, respectively

[^18]:    ${ }^{7}$ We apply the Heron formula to the triangle $\triangle A B C$ with sides of different length

[^19]:    ${ }^{8}$ There are the following quadrilaterals: square, rectangle, trapezoid, parallelogram, rhombus, deltoid, quadrilateral, any an inscribed circle and a circle encircled by a quadrilateral.
    ${ }^{9}$ The construction of a square with a compasses and a ruler is described in the Basic Figures project.

[^20]:    ${ }^{10}$ The theorem about a circle inscribed in a quadrilateral is described in paragraph o quadrilaterals This theorem applies to all quadrilaterals, including trapezoids

[^21]:    ${ }^{11}$ The theorem of a circle inscribed in a quadrilateral and inscribed in a quadrilateral is described in paragraph o quadrilaterals This theorem applies to all quadrilaterals in this deltoid

[^22]:    ${ }^{12}$ Here $\angle A B C$ is the angle at the vertex $B$ and the arms $[A, B]$ and $[A, D]$.

[^23]:    ${ }^{13}$ Here we use the equation of addition

[^24]:    ${ }^{14}$ Length of the vector product $|\vec{v} \times \vec{w}=|\vec{v}| *| \vec{w} \mid * \cos \alpha$, where $\alpha$ is the angle between the vectors $\vec{v}, \vec{w}$. The quadrilateral field $P_{A B C D}=|\vec{v} \times \vec{w}=|\vec{v}| *| \vec{w} \mid * \cos \alpha$ is equal to the length of the vector product vctor.
    ${ }^{15}$ The area of a concave quadrilateral is equal to the difference of vector products $P_{A B C D}=\frac{1}{2} \vec{v} \times \vec{w}-\frac{1}{2} \vec{Q} \times \vec{t}$

[^25]:    ${ }^{16}$ The length of the vector $\vec{v}=\left[v_{1}, v_{2}, v_{3}\right]$ with the coordinates $v_{1}, v_{2}, v_{3}$ is given by the formula $|\vec{v}|=$ $\sqrt{v_{1}^{2}+v_{2}^{2}+v_{3}^{2}}$

[^26]:    ${ }^{17}$ Elementary constructs described in previous paragraphs

[^27]:    ${ }^{1}$ Coordinates $v_{1}, v_{2}, v_{3}$ of a free vector $\vec{v}=\left[v_{1}, v_{2}, v_{3}\right]$, we write in square brackets.
    ${ }^{2}$ Free vector does not depend on the position location on the plane or in space.

[^28]:    ${ }^{3}$ Scalar product or dot product of vectors, it is a number

[^29]:    ${ }^{4}$ The scalar product $(\vec{v}, \vec{w})=0$, if and only if $\vec{v} \perp \vec{w}$, in symbols we write

[^30]:    ${ }^{5}$ The length of the vector $\vec{v}=\left[v_{1}, v_{2}, v_{3}\right]$ with the coordinates $v_{1}, v_{2}, v_{3}$ in the Cartesian space $R^{3}$ are given by the formula $|\vec{v}|=\sqrt{v_{1}^{2}+v_{2}^{2}+v_{3}^{2}}$

[^31]:    ${ }^{1}$ In higher mathematics trigonometric functions are defined by power series

[^32]:    ${ }^{2}$ The domain of the $f(x)$ function is the set of $x$ arguments for which $f(x)$ is defined
    ${ }^{3}$ Identity that is, the equality that holds for all $x$ values in the identity domain $x \in D$.

[^33]:    ${ }^{4}$ An odd multiple of a right angle $(2 \pi+1) \frac{\pi}{2}, \quad k=, \pm 1, \pm 2, \pm 3, \ldots$;
    ${ }^{5}$ An even multiple of a right angle $2 k * \frac{\pi}{2}=k \pi, \quad k=0, \pm 1, \pm 2, \pm 3, \ldots ;$

[^34]:    ${ }^{6}$ Index $k \in C=\{0, \pm 1, \pm 2, \pm 3, \ldots:\}$ runs the whole set of integers.

[^35]:    ${ }^{7}$ Remember that $|A D|$ is the length of the segment $[A, D]$

[^36]:    ${ }^{1}$ More general concept of probability then Laplace's one, is stated in the axiomatic definition which includes of equally and not equally probables elementary events. Axiomatic concept of probability is beyond scope of this text.

[^37]:    ${ }^{2}$ Two combinations or subsets are different if and only if they differ with at least one element

